GRAPHIC ALGEBRA.

AN ELEMENTARY TEXT BOOK FOR COLLEGE STUDENTS.

GRAPHS.

BY

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PREFACE.

The value of graphic methods is so universally recognized that it is unnecessary to apologize for them, or to enter into any argument for their use.

How far in the crowded condition of our college courses, the graphic method should be allowed to encroach upon or supersede the methods of purely abstract reasoning, must be decided by each instructor for his own class room, and in consideration of his own peculiar object to be attained.

This text book embodies the notes used in the author's class room from time to time.

The subject is introduced with a brief description of rectangular co-ordinates of two and three dimensions, followed by a geometrical interpretation of $\sqrt{-1}$.

In chapter IV an attempt has been made to supply a concrete conception of the appearance and disappearance of real and imaginary roots, and of the close connection between the two kinds, so as to enable the student to answer for himself—"What becomes of the real roots when they disappear, and whence do they reappear?" This is done by considering x as a complex quantity, and $\sqrt{-1}$ as a quadrantal versor, thus introducing a new ordinate perpendicular to the plane of the other two; $\sqrt{-1}\,x$ corresponding to $-y$, and $\sqrt{-1}\,z$ to $+y$.

This gives a system of co-ordinates in space and a corresponding surface; in the case of equations of the second degree, a hyperbolic paraboloid, and for higher degrees, more complex surfaces.

The intersection of a horizontal (i. e. perpendicular to the z axis) plane through the origin with this surface will give a curve whose z ordinates will in general have all their real parts zero, but the imaginary parts not zero, that is z will be a pure imaginary. But there are certain points in this section in which the z ordinate will be entirely zero, both its real and imaginary parts having disappeared. The locus of these points on the surface is a curve or curves in space, the intersection of which with a horizontal plane through the origin will give the root points of the given curve, either real or imaginary points.

Being thus brought before the student in the shape of concrete lines and points, he is able to see at a glance the relation between the two kinds of roots. He has moreover the advantage which a geometrical conception always gives over one arising from purely abstract reasoning. He has a physical illustration, as it were, that the two kinds of roots are one and the same thing, differing only in location; that the disappearance of real roots is not annihilation, but merely change of relation. This is impressed on the student by dropping the antiquated and misleading terms real and imaginary, and using the more appropriate terms *non i* and *i*.

In chapter VII is shown the method of plotting the the *non i* and *i* root curves in pseudo perspective. This was necessarily preceded by a short chapter on the

General Theory of Equations in order to prepare the student for the numerical calculations. Only the simple theorems are used, stated seriatim, and are assumed as already known by the student from other text books. This includes a section on Derived Polynomials in which the student is introduced to some of the geometrical properties of these functions.

In chapter VIII, the general formulæ for the nth roots of ± 1, are arrived at by easy steps from previous geometrical considerations, without the introduction of Demoivre's Formula, or of the exponential form of the complex quantity.

The next two chapters discuss complex quantities, and the Argand Diagram in which are given some of the simple principles of the calculus of complex quantities and some simple complex functions.

The geometrical rules for complex multiplication and division constitute a novel feature which will do much toward simplifying the graphical representation of complex functions. The student cannot become too thoroughly familiar with these rules.

The course closes with the discussion of the simpler Riemann Surfaces. The subject has been presented in as simple a manner as possible, and the connection of the surfaces made to depend upon the number of closed curves, rather than upon "through sections"*. The latter method has the advantage of being more general but is not so well adapted to an elementary text book. This puts before the English speaking student for the first time, so far as the author is aware, a more or less complete text book on the elementary theory of

*Querschnitt.

Complex Functions and the Riemann Surfaces. The close connection between the curves in the Argand Diagram and upon the Riemann Surfaces has been more explicitly shown than is usually the case and the fact impressed upon the student that one is merely a modification of the other, and not a radical change of method. The surfaces delimited by closed curves are also noticed. A thorough knowledge of these chapters will do much to help the student over the first difficulties in the Function Theory.

CONTENTS.

GRAPHIC ALGEBRA.

CHAPTER I.

PLANE CO-ORDINATES.

The position of a point in a plane is determined by its distances from two perpendicular lines called co-ordinate axes. This is shown in Fig. 1, where the position of the point P is definitely determined as soon as we know the distances oz, ox. The indefinite lines oz, ox are called the *co-ordinate axes*; ox *the* x–axis, and oz the z–axis. The intersection o is called the *origin*. The distance ox is called the *abscissa* of the point and is denoted by x, and the distance oz is called the *ordinate* of the point P, and is denoted by z. They are measured *from* the point o.

Abscissa are considered positive to the right, and ordinates positive upward; and negative in opposite directions.

The abscissa and ordinate taken together, are sometimes called co-ordinates of the point.

EVERY EQUATION BETWEEN TWO VARIABLES, MAY BE CONSIDERED AS REPRESENTING SOME LINE, EITHER STRAIGHT OR CURVED, IN WHICH ONE VARIABLE REPRESENTS THE ORDINATES AND THE OTHER THE ABSCISSAS.

To illustrate this take the equation $z=2x+4$. In Fig. 2 let OZ and OX be the co-ordinate axes. Along

the axis OX take the different values of x, 01, 02, 03, etc , and from these points erect the perpendiculars 1*a*, 2*b*, 3*c*, etc. The lengths of these perpendiculars are found from the given equation by substituting the different assumed value of x, and calculating the corresponding values of y. Thus, for $x=1$, 2, 3, 4, etc., we find respectively $z=6$, 8, 10, 12, etc. Having plotted the points a, b, c, d, etc., we connect these points by the line *abcd*, and this line will be the geometrical representation of the given equation. It is the locus of the point which has x and z for its ordinates.

Negative values of x, give corresponding value of z, thus: $x=0, -1, -2, -3,$ etc give $z=4$ 2, 0, -2, etc.

The unit 01, may be any convenient unit, ⅛ inch, ½ inch, etc. The unit along the axis of z may be taken the same as that along the axis of x, or it may be taken a different unit, just as in the latitude of New York. the unit of longitude is $1^\circ=52.5$ miles and the unit of latitude quite different, viz: $1^\circ=69.1$ miles.

The use of cross section paper, or in the case of different units of length, of profile paper, will save much measuring.

In evaluating the different expressions, for certain values of the unknown quantity, it is a saving of time to make use of the method of synthetic division which is described below.

SYNTHETIC DIVISION.

Let it be required to divide $x^4+Ax^3+Bx^2+Cx+D$ by $(x-a)$. Performing the operation by long division,

NOTE.—The locus of an equation is the position of all the points (generally a continuous line) whose co-ordinates satisfy the given equation.

we get the quotient $x^3+(a+A)x^2+(a^2+Aa+B)x+(a^3+Aa^2+Ba+C)$ and the remainder $a^4+Aa^3+Ba^2+Ca+D$.

Now this operation can be performed much more rapidly as follows: Write the co-efficients (called detached because they are separated from the variables) in a horizontal line as below, with a at the right hand in the form of a divisor.

$$\begin{array}{ccccc|c} 1 & +A & +B & +C & +D & a \\ & a & a^2+Aa & a^3+Aa^2+Ba & a^4+Aa^3+Ba^2+Ca & \\ \hline & a+A & a^2+Aa+B & a^3+Aa^2+Ba+C & a^4+Aa^3+Ba^2+Ca+D & \end{array}$$

Then multiply the first co-efficient by a and write the result under the second, and add, giving $a+A$. Multiply this sum by a, write the product under the third co-efficient, and add, giving a^2+Aa+B. Continue this alternate multiplying and adding until all the co-efficients (including the absolute term, which is the co-efficient of x^0) have been used, and we obtain $a^4+Aa^3+Ba^2+Ca+D$ for the last sum. By comparing this with the results of the long division, we see that, neglecting the last sum, the first co-efficient and the other sums taken in order, are the co-efficients of the quotient when arranged according to the descending powers of x.

The last sum is evidently the value of the original expression when a is put in place of x. Hence, TO EVALUATE $f(x)$ FOR ANY PARTICULAR VALUE OF x, AS $x=a$,—*arrange the co-efficients of $f(x)$ in order, (supplying the zero co-efficients when necessary, so as to have a co-efficient for every power of x); multiply the first co-efficient by a, add the result to the second, multiply the sum by a,*

add the result to the third, multiply by a, and so on. The last sum will be the required value of $f(a)$.

TO DIVIDE $f(x)$ BY $(x-a)$. *Proceed as in the rule for evaluation; the last sum will be the remainder. The other sums together with the first co-efficient, when, commencing at the right, they are multiplied by the different powers of x, commencing with* x^0, *and the results added, will give the quotient.*

If the last sum (i. e. the remainder) is zero, $(x-a)$ divides the given $f(x)$ exactly.

Draw the lines represented by the following equations:

1. $z=-2x-3$.
2. $z=x+4$.
3. $z=4x+2$.
4. $z=-x+3$.
5. $z=3x-2$.
6. $2x-3z=4$.
7. $\frac{1}{3}(x-2z)=4$.
8. $z=x^2-3x+2$.

Construct the locus of the equations:

9. $z=x^2+x-3$.
10. $z=x^2-4x+2$.
11. $z=x^2-3x+6$.
12. $z=x-\frac{1}{4}x^2+3$.
13. $z=\dfrac{x}{1+x^2}$.
14. $z=x^3-\frac{7}{2}x^2+2x+2$.
15. $z=x^3-6x^2+11x-6$.
16. $z=x^4-9x^2+4x+12$.
17. $z=x^4+2x^3-3x^2-4x+4$.
18. $z=x^3-2x-5$.

The student will possibly have noticed by this time that the locus of every equation of the first degree is a straight line. This being the case, it is of course necessary to find only two points of the line and these will determine the line. The two most convenient points are the points of intersection with the axes. Thus in example 1, making $z=0$, we get $x=-\frac{3}{2}$ which gives, the point where the AXIS of X is crossed. Making $x=0$,

we get $z = -3$, the point where the AXIS OF Z is crossed. Drawing a straight line between these two points, we have the required locus.

Plot examples 1 to 18 making the unit along the AXIS OF Z $\frac{1}{3}$ of that along the AXIS OF X. It will be seen that the only change is the flattening or elongation of the curve: the character of the curve is not changed.

In an equation between two variables the value of x which makes z equal to o, is called a root of the equation.

From this definition it is evident that to find the root of an equation graphically it is simply necessary to ascertain the value of x for the point where the curve crosses the x axis. These points we will call the *root points.*

Thus in the equation $z=x^4-20x^2+64$, the curve crosses the *axis of abscissas* at the points, -4, -2, $+2$, $+4$, hence the 4 roots are -4, -2, $+2$, $+4$. See Fig. 3.

In finding the roots graphically, only that part of the curve, of course, which lies near the AXIS OF X need be plotted, and the scale used can be quite large, thus giving greater accuracy.

Example.—Construct graphically the roots of the equation $x^2-3x-2=0$. Placing this expression equal to z we have $z=x^2-3x-2$. We find that the curve crosses the AXIS OF X between 3 and 4 and also between -1 and 0. Hence a root of the equation lies between -1 and 0; and also between 3 and 4.

First to find the positive root. For $x=3\frac{1}{2}$, $3\frac{3}{4}$ we find the corresponding values of $z=-\frac{1}{4}$, $\frac{13}{16}$. Drawing the curve through these points we find that it crosses the AXIS OF X at about 3.56 which is accordingly one root of the equation.

Secondly, for the negative root. For $x=-\frac{1}{2}, -\frac{3}{4}$ we have $z=-\frac{1}{4}, \frac{18}{16}$. Drawing the curve through these points we find that it crosses the axis at $-.56$, which is accordingly the negative root of the equation.

Construct the roots of the equations,

(1) $x^2-8x=14$. (4) $x^3-7x+7=0$.

(2) $x^3-12x^2+36x=7$. (5) $x^4-12x^3+50x^2-84x=49$.

(3) $x^3-x^2-10x+6=0$. (6) $x^3-4x^2-11x+43=0$.

MOVING THE ORIGIN UP AND DOWN.

If the equations (which differ only in the absolute term.)

$$y=x^2-4x-5$$
$$y=x^2-4x+3$$

be plotted we get the following figure, in which the two curves are exactly the same shape, and differ only in being raised or lowered on the AXIS of z. It will also be seen that for any one value of x, the ordinates of the two curves differ by the difference between the absolute terms. Thus for $x=0$ we have $z=-5$ for one curve and $+3$ for the other, that is the points of one curve will be 8 units above the corresponding points of the other curve.

The upper curve is evidently simply the lower curve after the origin has been lowered 8 units. Hence the following

RULE.—*If the absolute term of an equation be increased by m units, the origin must be moved downward m units, and vice versa.* CONVERSELY, *if the origin be moved up or down m units, the absolute term must be respectively decreased or increased by m units.*

Thus we see that having the graph* of an equation given, we can put the AXIS of X anywhere we please without altering the *form of the curve*; the only effect being (so long as the AXIS of Z is not moved) to change the *absolute term* of the equation as the X AXIS is moved up and down, and also the *root points*, and therefore the *roots* of the equation.

CHAPTER II.

CO-ORDINATES IN SPACE.

Having shown that an equation between two variables represents a line, we will now show that an equation between three variables represents a surface.

In this case instead of the two co-ordinate axes only, we have three co-ordinate axes at right angles to each other, and the three co-ordinates measured along the axis, viz: x, y, z.

The three mutually perpendicular axes are shown in perspective in Figure 5, with the co-ordinates of a point. In representing the perspective, it will be found convenient to lay off the y axis at about 40° with the x axis as represented.

Before going further, we will discuss the method of representing the surface whose co-ordinates are given by an equation. If we wished to represent the surface of an apple by lines only and more fully than a mere outline would do, we could show the curves made by the intersection of the surface with parallel planes, as shown in Fig. 6.

**I. E.*– The locus or graphical representation of the equation.

In order to show the whole section and not merely the part towards the observer and in order that the curves may not intersect each other, we may imagine the solid stretched along the axis of x, or what is equivalent take the unit along x larger than the other axis units.

Thus, omitting the outline of the apple, we get Fig. 7. In the same way we could represent any solid. To get these sections,* we suppose x to be constant for each section and y and z to vary. The graph traced by y and z for that special value of x will of course be the outline of the section of the surface made by a plane perpendicular to x, and at a distance from the origin equal to the value of x.

That portion of a plane and a sphere lying in the first angle would, in this manner, be represented as shown in Figures 8 and 9.

EXAMPLES.

(*a.*) Show the surface of the cone $y=\left(\dfrac{10-z}{2.1}\right)^2-x^2$ by drawing the graphs† for the cases where $x=0, 1, 2, 3$, making the x units 4 times the y and z units. (See Figures 10, 11.)

(*b.*) Show the surface of the sphere $x^2+y^2+z^2=25$ by means of the graphs for the 3 cases $x=0, 1, 2$.

*These sections are called the contour lines of the graphic surface. Chrystal. I have however, taken them vertical, instead of horizontal, for reasons which will be apparent further on.

†These graphs should be shown between the points where they pierce the co-ordinate plane yx.

(*c.*) Show the plane $\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=1$ by means of the graphs for $x=0, 1, 2, 3, 4, 5$*.

(*d.*) Show the surface $\frac{x^2}{9}-\frac{y^2}{2}-\frac{z^2}{7}=1$.

CHAPTER III.

IMAGINARIES.

INTERPRETATION OF $\sqrt{-1}$.

From simple algebraic principles we have

$$\text{“}(\sqrt{-1})(\sqrt{-1})=-1=\sqrt{-1}\times\left\{\sqrt{-1}\times 1\right\}.$$

Consequently $\sqrt{-1}$ must satisfy the condition that if twice successively applied to $+1$ by the process of $\times$ [whatever that be] it has the effect of changing $+1$ into -1.” (De Morgan. Double Algebra.)

Since if $+x$ is represented by direction to the right and $-x$ by direction to the left, -1, by which, if x be multiplied it will become $-x$, may be considered as a GEOMETRICAL OPERATION whose mode of operation we may perhaps at present not understand, but the *result* of which we can see. Thus $+x$ is represented by OA in Fig. 12. Performing on this line, the geometrical operation (-1), it becomes $OB=-x$.

Now if multiplying x by -1 (or $\sqrt{-1}\sqrt{-1}$) turns it around the point O through the straight angle BOA, viz: 180°, multiplying it by $\sqrt{-1}$ must turn it through

*These graphs should show only those portions lying between the points where they pierce the co-ordinate planes.

half this angle. to the position OC, since multiplying it by $\sqrt{-1}$ twice has the effect of turning it through all the angle. Thus $\sqrt{-1}\sqrt{-1}\,x$ becomes $-x$ or OB, hence $\sqrt{-1}x$ must be OC.

Hence multiplying a geometrical line by $\sqrt{-1}$ may be considered as equivalent to revolving it 90° around the origin*.

Applying this to the co-ordinate axes we have the set shown in Fig. 13. We are now prepared to interpret geometrically such expressions as

$$z=x+\sqrt{-1}\cdot 3x.$$

$\sqrt{-1}x$ and $\sqrt{-1}z$ evidently, merely take the place of the y co-ordinate of Chapter II, $\sqrt{-1}x$ taking the place of $-y$, and $\sqrt{-z}$ that of $+y$. The surface represented by any equation between the three variables z, y, and $\sqrt{-1}x$, $\sqrt{-1}z$ would then be represented exactly in the same manner as in Chapter II.

IMAGINARY ROOTS.

Strictly speaking, this name is misleading, for it leads the student to suppose that they have no existence, which is not the case by any means, as we shall see further on. The student should understand definitely that they are actual existent roots just as the real ones are, but that they are roots which include the expression $\sqrt{-1}$.

The student will readily see that the question, what are imaginary roots will be answered as soon as we are able to determine the significance of the symbol $\sqrt{-1}$. This we have already done, and we are now prepared to investigate imaginary roots.

* We have a right of course to put any interpretation on this expression that we please so long as it does not conflict with any previous conditions or definitions; and it will be found that this interpretation is entirely consistent with all previous definitions.

CHAPTER IV.

NON i ROOTS OF SECOND DEGREE EQUATIONS.

We will now take up equations of the second degree. For example, in the equation $z=\xi^2+2\xi+2$, let us suppose ξ to have both real and imaginary values. To find the values of z, we give all possible values to ξ both real and imaginary, and compute the corresponding values of z. If we put the equation in the form $z=(x+n\sqrt{-1})^2+2(x+n\sqrt{-1})+2$ found by putting $\xi=x+n\sqrt{-1}$ in which x and n are real numbers, we can perform the numerical calculations more systematically.

Considering these values as co-ordinates, we have three mutually rectangular co-ordinates, viz., x, y, and $\pm\sqrt{-1}x$ which correspond to $\mp y$, and $\pm\sqrt{-1}z$ which correspond to $\pm y$, as shown in Fig. 13.

Taking all values of n and substituting, and plotting the result (as shown hereafter) we get a saddle-shaped surface, something like a high pass between two mountains, or like a high Mexican saddle. This is represented in Fig. 14. That portion of the solid in front of the co-ordinate planes, is outlined in heavy lines.

We will now proceed to plot this solid, or rather the sections of it by vertical planes.

Preparatory to so doing we will tabulate the values of z, as follows: for 4 sections at the distance, -2, -1, 0 and $+1$ from the origin, writing i instead of $\sqrt{-1}$, for the sake of brevity.

$z=\xi^2+2\xi+2.$

ξ $x+ni$	z	ξ	z	ξ	z	ξ	z
$-2-5i$	$-23+10i$	$-1-5i$	-24	$0-5i$	$-23-10i$	$1-5i$	$-20-15i$
$-2-4i$	$-14+8i$	$-1-4i$	-15	$-4i$	$-14-8i$	$1-4i$	$-11-12i$
$-2-3i$	$-7+6i$	$-1-3i$	-8	$-3i$	$-7-6i$	$1-3i$	$-4-9i$
$-2-2i$	$-2+4i$	$-1-2i$	-3	$-2i$	$-2-4i$	$1-2i$	$1-6i$
$-2-i$	$+1+2i$	$-1-i$	0	$-i$	$1-2i$	$1-i$	$4-3i$
-2	$+2$	-1	1	0	2	1	5
$-2+i$	$+1-2i$	$-1+i$	0	$+i$	$1+2i$	$1+i$	$4+3i$
$-2+2i$	$-2-4i$	$-1+2i$	-3	$+2i$	$-2+4i$	$1+2i$	$1+6i$
$-2+3i$	$-7-6i$	$-1+3i$	-8	$+3i$	$-7+6i$	$1+3i$	$-4+9i$
$-2+4i$	$-14-8i$	$-1+4i$	-15	$+4i$	$-14+8i$	$1+4i$	$-11+12i$
$-2+5i$	$-23-10i$	$-1+5i$	-24	$+5i$	$-23+10i$	$1+5i$	$-20+15i$

From this table we proceed to plot the vertical sections of the solid, exactly as in the case of an ellipsoid or sphere, except that $+ix$ and $-iz$ take the place of $-y$, and $+iz$ and $-ix$ take the place of $+y$.

Thus for the values $x=-2-i$ we get $z=1+2i$ which would be plotted as shown in Fig. 15, and for $x=-2-2i$ $z=-2+4i$ which would give Fig. 16. These put in one figure would give Fig. 17. We thus get the Fig. 18, in which the imaginary portions of z are indicated by heavy lines. To avoid confusing the figure, the co-ordinates of only one side of each curve are shown.

The advanced student will recognize this as the surface called the Hyperbolic Paraboloid.

Upon inspection of this figure, the student will easily see that of all the sections perpendicular to the AXIS of X, the one through CD [see Fig. 19] is the only one that has its z ordinates all real; all the other sections have the z ordinates partly *real* and partly *imaginary*.

In the same manner, considering sections made by planes perpendicular to the axis of iz, the only section not having the ordinates z and x partly real and partly imaginary, is the section through the axis of z; and this section, it is very evident has both its z and x ordinates all real.

To find the values of x corresponding to $z=0$, that is, the roots of the given equation, we pass a horizontal plane through the origin, which of course intersects the surface [or the vertical sections of the surface] in all the points which correspond to $z=0$, (as well as in many points in which z is not equal to zero but to iz, *i. e.*, the real part of z has disappeared, but an imaginary part still remains).

This is shown in Fig. 19, where the curve ADB and the corresponding curve on the other side EFG show the intersections of the horizontal plane through the

origin with the surface. By comparing this figure with Fig. 18 it will readily be seen that D and F are the only points on these curves which have their z ordinate equal to zero, *i. e.*, in which both the real and imaginary parts of z have disappeared. For all the other points, the *real* part of z has disappeared, but not the *imaginary* part.

Hence, for the points D and F, the z ordinate has entirely disappeared, and the x ordinate is partly real and partly imaginary. In other words the values of x which reduces z to zero (*i. e.* satisfies the equation) is

$$x = \text{the real part OK} \pm \text{the imaginary part} \left\{ \begin{matrix} \text{KD} \\ \text{KF} \end{matrix} \right.$$

If the origin O. had been in the plane of DCF, the values of x which made $z=0$, would have been entirely imaginary (*i. e.* would have had no real part along the axis of x).

Now if we diminish the absolute term of our equation by 5, we get

$$z = x^2 + 2x - 3.$$

This, as we have already shown, merely amounts to raising the co-ordinate centre 5 units. Now passing a horizontal plane through this new centre we get Fig. 20, in which the dotted lines show the curves, etc., of the previous figure. Evidently the only point where $z=0$ (*i. e.* both real and imaginary parts are zero) is at the points M and N.

By comparing these figures, we see that as the horizontal plane is lowered, the zero points [*i. e.* the root points] approach each other, until at the point C the two root points coincide, and as the plane is lowered

from that point, they pass down the *i* curves* CD, CH, and we have what are termed *i* (misleadingly called imaginary†) roots.

As the plane is raised the *i* roots pass up the curves DC, HC, until they coincide at the point C, and thence they pass up the *non i* curves CM and CN, giving *non i* (generally called real) roots.

The student can now see why some equations have *non i* and others *i* roots; also how the one kind passes into the other; how the disappearance of one kind is succeeded by the appearance of the other.

Having now ascertained the characteristics of the contour sections, we see that we need pay no attention to anything but the *non i* and *i* curves, that is to say, the curves whose z ordinates are entirely real; of these the one whose x ordinates are also entirely real is the *non i* curve, and the other is the *i* curve.

And furthermore, we can neglect the absolute term and plot the curve with the origin at the intersection of the two curves and then, by raising or lowering the axis of x, introduce the absolute term. Having done this we are ready to measure off on the axis of x the *non i* roots, or upon the ix axis, the *i* roots.

It will be noticed that we have two *non i* roots, or two *i* roots, according to the height at which we pass the horizontal plane.

* From here on, the imaginary roots and imaginary curves will be designated as the *i* roots, and *i* curves.

† The terms real and imaginary are misleading to the student. A better nomenclature would be *non i* and *i*. These names would impress upon the student the fact that he was dealing with really existent roots, though different from the ordinary or *non i* ones, and not mislead him into thinking that imaginary roots, so called, have no existence.

If the horizontal plane is tangent to the curves, we have *equal* roots, and at this point the *non i* and *i* roots have one and the same value, viz., $x+0\sqrt{-1}$, which is obviously either *non i* or *i* according as we disregard or not, the term $0\sqrt{-1}=0 \cdot i$.

CHAPTER V.

EQUATIONS OF THE THIRD DEGREE.

The general equation of the 3d degree is of the form $y=x^3+ax^2+bx+c$ which can always be reduced to the form $y=x^3+nx+m$ by putting $x-\frac{a}{3}$ in place of x. This will not change the form of the curve. Since we may neglect the independent term, so far as the form of the curve is concerned, we may take as the general form of the equation, $y=x^3+nx$ which gives three general types according to the value of n, viz :

$$y=x^3-nx$$
$$y=x^3$$
$$y=x^3+nx$$

Plotting the contour sections of the surfaces of which these are the equations, in the same manner as we did in the case of equations of the second degree, we get the surfaces shown in Figs. 21–25, in which the *i* curves are drawn in heavy lines.

GENERAL RESULTS. By comparing the results of the two preceding chapters we see that the only portions of the surface which need take up our attention are traces thereon of the *non i* and the *i* curves.

It will be remembered that in the equation of the 2d degree we had two *non i*, two equal or two *i* roots, according to the height at which we passed the horizontal cutting plane.

Likewise, by comparing Figs. 21 to 25 we see that we can get three *non i*, or two *i* and one *non i* roots, according to the height at which we pass the horizontal plane. But the following special cases must be noticed. In Figs. 21, 22 we can get two *i* and one *non i* or three *non i* roots. In Figs. 23 to 25 we can get only one *non i* root, and the other two must be *i* roots.

Comparing Figs. 21–25, we see that as we pass from $x^3—nx$ through x^3 to x^3+nx, the *i* curves begin as two separate curves, which gradually approach each other and finally coalesce. The *non i* curve begins with two very decided elbows, which get flatter as we proceed and finally the elbows disappear altogether for x^3, and the *non i* curve begins to approach a straight line as the values of n increase.

Before proceeding to show how to plot these principal contours, or *i* and *non i* curves as we call them, we will in the next chapter take up the subject of the general theory of equations. This will give the means of plotting these curves much more easily and expeditiously, than we could do without the principles found there.

CHAPTER VI.

GENERAL THEORY OF EQUATIONS.

We will give here a short statement of the general theory of the roots of equations, so far as we may need the principles in the following chapters.

We shall assume at the start certain principles, the proofs of which are to be found in all text books on Algebra, and exemplify here merely the practical use of these principles in the discussion of algebraic graphs. We consider equations of one unknown quantity only.

GENERAL PRINCIPLES.

(*a*) Every equation of the nth degree has n roots and no more.

(*b*) If a is a root of the equation $F(x)=0$, then $F(x)$ is divisible by $x-a$; and conversely, etc.

(*c*) If a is a root of the equation $F(x)=0$, then

$$\frac{F(x)}{x-a}=0$$

is an equation of the next lower degree, the roots of which are the remaining roots of the given equation.

(*d*) In an equation of the form

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots\ldots\ldots p_n=0.$$

p_1=Sum of the roots with their signs changed.

p_2=Sum of the products of the roots taken two at a time.

p_n=Product of the n roots with their signs changed.

(*e*) Imaginary roots occur in pairs.

(*f*) Changing the alternate signs of a complete equation, changes the signs of all the roots.

The last statement can be shown graphically as follows: Take for example the equation of the second degree $x^2+nx=z$. The final ordinate z can be considered as made up of the sum of two partial ordinate, viz. $z_1=nx$ and $z_2=x^2$. By plotting these partial ordinates (thus getting two curves) and then adding the corresponding ordinates we combine the two curves into one. This is shown in Fig. 25* where the two curves $z_1=nx$ and $z_2=x^2$ are shown in light lines and $z=z_1+z_2$ is shown by a heavy line.

Now suppose we change the alternate signs in the equation. This gives us $z=-x^2+nx=z_1-z_2$. The result of subtracting this second ordinate instead of adding it, is shown in Fig. 25**. If our equation had had an absolute term, say $-m$, the only difference would have been to move the x axis up m units in Fig. 25* as shown by the heavy line, and to move it down m units in Fig. 25** (since m changes its sign for this figure). Now comparing the two figures, the root points being indicated by small circles, we see that the roots in one figure are the negative of the roots in the other, a result produced by changing the alternate signs in the original equation.

We might have arrived at the same result as follows: Changing the alternate signs produces exactly the same result as subtituting $-x$ and $-z$ for x and z. But changing our x into $-x$ and z into $-z$, simply transfers the curve from one quadrant to the diagonally opposite one as shown in Fig. 25*** where the light curve becomes the heavy one by changing the signs of both ordinates.

These same considerations can be extended to equations of higher degrees with the same results.

DERIVED POLYNOMIALS OR DERIVATIVES.

If in the general equation of the n^{th} degree

$$f(x)=x^n+Ax^{n-1}+Bx^{n-2}+\;.\;.\;.\;.\;.\;.\;.$$

we substitute $(x+a)$ for x, we get

$$f(x+a)=(x+a)^n+A(x+a)^{n-1}+B(x+a)^{n-2}+\;.\;.\;.\;.\;.\;.$$

which after expanding by the binomial theorem, and arranging according to the ascending powers of a, becomes,

$$f(x+a)=$$

$$\begin{array}{r|r|r|l}
x^n & +\quad nx^{n-1} & a+\quad n(n-1)x^{n-2} & a^2+\;.\;.\;.\;. \\
+Ax^{n-1} & +(n-1)Ax^{n-2} & +(n-1)(n-2)Ax^{n-3} & \\
+Bx^{n-2} & +(n-2)Bx^{n-3} & +(n-2)(n-3)Bx^{n-4} & \\
\vdots & \vdots & \vdots &
\end{array}$$

which may be written

$$f(x+a)=f(x)+f'(x)a+f''(x)\frac{a^2}{\underline{|2}}+f'''(x)\frac{a^3}{\underline{|3}}+\;.\;.\;.\;.\;.$$

$f(x)$ is called the primitive, $f'(x)$ is called the first derived polynomial or derivative, $f''(x)$ is called the second derived polynomial or derivative, and so on.

It will be seen that the primitive is the original function. The first derivative is derived from the primitive according to the following:

RULE.—*Multiply each term by the exponent of x in that term, and then diminish the exponent of x by unity.*

The student will notice that each derived polynomial or derivative is derived from the preceding one by the same rule.

EXAMPLES.

Ex. 1. What are the successive derivatives of x^3-7x^2+8x-3? Ans. $3x^2-14x+8$, $6x-14$, and 6.

Ex. 2. What are the successive derivatives of $x^4-8x^3+14x^2+4x-8$? Ans. $4x^3-24x^2+28x+4$, $12x^2-48x+28$, $24x-48$, 24.

Ex. 3. What are the successive derivatives of $x^3-3x^2-9x+27$? Ans. $3x^2-6x-9$, $6x-6$, 6.

THEOREM.—*The numerical value of the first derivative for any value of $x=a$, is the tangent of the angle which the graph makes at that point with the axis of x.*

In Fig. 26 $f(x)=$BA, $x=$OA, $f(x+a)=$CE, $x+a=$OC, BE=secant, BF=tangent, $\varphi=$angle EBD, $\theta=$angle FBD=angle which the tangent to the curve at the point B makes with the axis of x.

From the figure we easily get

$$\frac{f(x+a)-f(x)}{a}=\frac{\text{CE}-\text{CD}}{\text{AC}}=\frac{\text{ED}}{\text{AC}}=\tan\varphi.$$

The student should call to mind the definition of a tangent, viz., A tangent is the limiting position of a secant, *i. e.*, the special case of a secant which passes through two coincident points.

Now as a grows smaller the point E approaches the point B and the line BE approaches the tangent BF,

and finally just as a disappears the two points and the two lines respectively coincide, and the angle φ becomes the angle θ. But

$$\tan\varphi = \frac{ED}{BD} = \frac{f(x+a)-f(x)}{a}$$

$$= f'(x) + f''(x)\frac{a}{\lfloor 2} + f'''(x)\frac{a^2}{\lfloor 3} + \ldots .$$

Hence as a disappears $\tan\varphi$ becomes

$$\tan\theta = f'(x)$$

all the other terms disappearing because they are multiplied by $a=0$.

Hence, the first derived polynomial measures the tangent of the angle of the inclination of the curve or graph at any point. Hence, to find the inclination of a curve at any point whose ordinates are x and y, find the first derived polynomial, and substitute the given value of x, the result will be the tangent of the required angle.

EXAMPLE.

Find the slope of the curve $y^2=4x$ at the point whose abscissa is 3.

Ans. $(\frac{1}{3})^{\frac{1}{2}}$

PROPERTIES OF THE GRAPH OF THE DERIVED POLYNOMIAL.

When $\theta=0$, or $180°$, that is when the curve of the primitive is horizontal, as it is at a maximum or minimum point, then $\tan\theta=0$.

Hence to find the maximum and minimum points, put the first derived polynomial (*i. e.* tan θ) $=0$, and the values for x which satisfy this equation will also be ordinates of the maximum and minimum points of the original equation.

Graphically, we plot the graph of the first derived polynomial and where this graph crosses the x axis will be ordinates of the maximum and minimum points of the primitive. (See Fig. 32.)

Since we are finding only the maximum and minimum points of the primitive we need not plot the whole graph of the derived polynomial, but only those parts near where it crosses the x axis.

EQUAL ROOTS.

Since the derived polynomial crosses the x axis at the maximum points we can easily see that if the x axis (either originally or by being moved up or down) passed through the maximum points (*i. e.* tangent to the curve) the derived polynomials must also cut the x axis at that point. If both curves touch the x axis at the same point (see Fig. 27) they must be satisfied by the same value of x, say a; hence both equations can be divided by $(x-a)$. That is, the primitive and the derived polynomial have a common divisor when the x axis passes through the maximum points. But when the x axis passes through the maximum point, the primitive curve has two equal roots.

Hence, *when an equation has two equal roots, there will be a common divisor between the equation and its first derived polynomial, and this common divisor will be of the form* $(x-a)$.

THREE EQUAL ROOTS.

If we take an equation of the fourth degree, with three equal roots and plot the primitive and derivative, we get a graph similar to Fig. 28, or to Fig. 29. An equation of the third degree with three equal roots would give Fig. 30 or if it had two equal roots, Fig. 31.

By comparing Figs. 27-31, it will be seen that when the derivative crosses the x axis with the primitive, we have two equal roots, and if a is the point of crossing, $(x-a)$ must divide the derivative, and $(x-a)^2$ the primitive, or $(x-a)$ is a common divisor of the primitive and derivative.

But if the derivative is tangent to the x axis as in Figs. 28-30, then the derivative must have two equal roots at that point, and the primitive must have three. Hence the common divisor must be of the form $(x-a)^2$.

EXAMPLES.

Ex. 1. Find the equal roots of $x^3-8x^2+21x-18$. Ans. 3, 3.

Ex. 2. Find the equal roots of x^4-6x^2-8x-3. Ans. -1, -1.

Ex. 3. Find the equal roots of $x^5-13x^4+67x^3-171x^2+216x-108$. Ans. 2, 2, 3, 3, 3.

Ex. 4. Find the equal roots of $x^3-x^2-8x+12$. Ans. 2, 2.

The student will find from his experience in plotting curves that every equation of the second degree will have a graph of the form shown in Fig. 27 for f', which will either cut the x axis in two separate points, two coincident points or two imaginary points.

For equations of the third degree, the student will likewise find that the graph starts in the lower left hand corner and after crossing the x axis an odd number of times, will end in the upper right hand corner.

This can also be shown analytically, but it would require a knowledge of the principles of Analytical Geometry.

These principles in regard to equations of the second and third degree illustrate the following

THEOREM.—*Just after the primitive passes a root, the primitive and its first derivative have the same sign, i. e, their graphs lie on the same side of the x axis.*

This is shown in Fig. 32, where it will be seen that just after passing the root points the primitive and its derivative are either both above or both below the x axis. This theorem can be shown analytically as follows:

Just after passing the root points, the primitive will have the form (see page 20)

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{\lfloor 2}+f'''(x)\frac{h^3}{\lfloor 3}+ \ldots\ldots$$

where h is a very small quantity. Now let r be a root of $f(x)=0$, and substitute in the above expression r for x; whence

$$f(r+h)=f(r)+f'(r)h+f''(r)\frac{h^2}{\lfloor 2}+f'''(r)\frac{h^3}{\lfloor 3}+ \ldots\ldots$$

Since r is a root, $f(r)=0$; and as h is very small the

terms containing the higher powers may be neglected without any appreciable error and we get

$$f(r+h)=f'(r)h, \text{ nearly, or } h=\frac{f(r+h)}{f'(r)} \text{nearly.}$$

Now since h is positive $f(r+h)$ and $f'(r)$ must have the same sign. Hence the theorem. Q. E. D.

ROLLE'S THEOREM—*Between two consecutive real root points of the primative, $f(x)=0$, there is always at least one real root point of the first derivative, $f'(x)=0$.*

Let a and β be two consecutive roots of $f(x)=0$, as shown in Fig. 32, then taking h very small, we have, as before

$$f(a+h)=f(a)+hf'(a)+\ldots=hf'(a) \text{ nearly,}$$

and $f(\beta-h)=f(\beta)-hf'(\beta)+\ldots=-hf'(\beta)$ nearly.

But $f(a+h)$ and $f(\beta-h)$ have the same sign, since there is no real root of $f(x)=0$ between a and β, hence $f'(a)$ and $f'(\beta)$ must have different signs, that is, the derivative must cross the axis between a and β. Hence the theorem, which might be enunciated as follows:

The derivative always crosses the axis between two consecutive root points of the primitive.

How do we know that $f(r+h)$ represents the function just after passing a root? What theorem could we similarly prove in regard to the primitive and its derivative just before the root points?

CHAPTER VII.

GENERAL METHOD OF PLOTTING THE *i* CURVES.

SOLUTION OF EQUATIONS OF THE FORM x^3+ax+b.

We first assume some real value for x and calculate the corresponding value of z, as shown in Fig. 33. It is evident from the figure, that for this value of z, we have two more values of x, each containing imaginary parts. These are indicated in the figure by the heavy lines. If we knew these values we could lay them off to scale and thus get points on the *i* curves.

To calculate these values, we divide out the root already found, which gives a new equation of the 2d degree, whose roots are the same as two of those of the original equation. This second degree equation is easily solved by the ordinary rules for quadratics.

Repeating this operation we can get as many points on the *i* curve as may be desired.

If the curve is of the form shown in Fig. 30, this process will always give points on the imaginary or *i* curves, but if the curve is of the form shown in Fig. 33, then it might happen that the value of x which we have assumed may throw the horizontal cutting plane [*i. e.* the plane whose height above the origin is determined by the value of z which corresponds to this assumed value of x] too low or too high to cut the *i* curves. In this case, of course, the other two values of of x would also be *non i* and we should get two points on the *non i* curve. The student can realize this by

drawing a horizontal line between the maximum and minimum points.

By refering to Fig. 33 it will be seen that the i curves do not lie inside of the maximum and minimum points of the *non* i curve but entirely outside. If we could determine the values of x corresponding to these points, we could easily confine our calculations to points on the i curves by merely taking the precaution of assuming values of x greater (and less algebraically) than those corresponding to the maximum and minimum points.

In practice, it would be well to determine the maximum and minimum points (if any) at first, and then proceed to calculate the co-ordinates of the i curves. To illustrate this we will take as an example $z=x^3-5x$; finding the first derivative, viz: $3x^2-5$, and placing it equal to zero, we find the values of x for the maximum and minimum points to be $x=\pm\sqrt{\frac{5}{3}}=\pm 1.291$.

Confining our attention to the maximum point and positive values of z, we get for the maximum point $x=-2.191$.

This is shown in Fig. 34. Now if we suppose the x axis raised until it is tangent to the curve at the maximum point (which is merely equivalent to adding an absolute term) we will have two equal *non* i roots at that point, each equal to -1.29.

To find the third value of x corresponding to this value of z we could calculate z for $x=-1.291$, substitute it in the given equation and then depress the equation to a quadratic by dividing by $(x+1.291)$, and then solve this quadratic equation, which would give us two

values of x, one equal to -1.291, the equal root already found, and the other the third root which we are in quest of, which in this case would be $+2.582$.

But an easier method for this case would be to put the sum of the three roots with their signs changed equal to the coefficient of the 2d term of the given equation, viz., $-a+2.582=0$, where a represents the third root which we seek.

This easily gives $a=2.582$ as before. Hence, if we wish to get points on the i curves, we must assume values of x greater than 2.582. Less values would give points on the *non i* curve. This is shown in Fig. 34.

Thus, assuming $x=2.6$ we get $z=4.575$ or $x^3-5x-4.575=0$. Dividing this by $(x-2.6)$ we get for the depressed equation, $x^2+2.6x+1.76=0$. Solving this we get $x=-1.3\pm.26\surd(-1)$.

For $x=2.7$ we get $z=6.18$, and for the depressed equation $x^2+2.7x+2.29=0$, whence the other values of x are $x=-1.35\pm68\surd(-1)$.

For $x=2.8$ we get $z=7.95$, and for the depressed equation $x^2+2\ 8x+2.84=0$, whence $x=-1.4\pm.99\surd(-1)$.

For $x=2.9$ we get $z=9.89$ and for the depressed equation $x^2+2.9x+3.41=0$, whence $x=-1.45\pm1.14\surd(-1)$.

Plotting these values we get Fig. 34.

For x^3-x we should get much the same figure, only the flexure of the *non i* curve would not be so great. And for x^3, we should get a *non i* curve of still less flexure, and one in which also the i curves have approached and intersected at the origin. This is shown in Fig. 35, where the ordinates are drawn quite closely together, thus optically determining a surface,

plane in the case of the x ordinates, and curved in the case of the ix ordinates. The intersection of these two surfaces is indicated by the dotted line, and the i curves are drawn heavy.

For $x^3+\frac{1}{10}x$ we have Fig. 36, and for x^3+x we should get a similar figure except that the curves would all have less flexure.

The student should plot carefully and to a large scale x^3-x and x^3+x, for as will be shown later, these can be used to solve the general equation of the third degree.

By examination of the graphs the student will easily convince himself that the i roots always occur in pairs.

SOLUTION OF EQUATIONS OF THE FORM

$$x^3+ax^2+bx+c=0.$$

By substituting $x=y-\frac{a}{3}$, the given equation reduces to $y^3\pm ny\pm m=0$ where m and n represent numerical values only. If in this we substitute $y=w\sqrt{(n)}$ we get

$$w^3\pm w\pm\frac{m}{n^{\frac{3}{2}}}=0.$$

The roots of this equation can be measured off on the plots of $x^3\pm x$, which we have already discussed, as follows: Find the value of $m\div n^{\frac{3}{2}}$ to two or three decimal places. Measure this value from the origin on the z axis, upward if minus, downward if plus. The distances to the *non* i or i branches of the curve meas-

ured from this point, will give the roots of the equation

$$w^3 \pm w \pm \frac{m}{n^{\frac{3}{2}}} = 0.$$

Substituting these values of w in the equation $y = w\sqrt{(n)}$ and then the values of y in

$$x = y - \frac{a}{3},$$

the results will be the roots of the original equation. If $n = 0$, our original equation, is transformed into $y^3 \pm m = 0$, whose roots can be found in a similar manner from the plot of $z = x^3$

CHAPTER VIII.

ROOTS OF THE EQUATION $x^n = \pm a$.

Equations of this form can always be written $x^n = a(\pm 1)^n$, whence $x = a^{\frac{1}{n}}(\pm 1)^{\frac{1}{n}}$. Hence, to solve $x^n = \pm a$, we simply need to multiply the roots of $x^n = \pm 1$ by $a^{\frac{1}{n}}$. This being the case, we will confine our attention to the form $x^n = \pm 1$.

In our discussion of equations of the second degree, Chap. IV., we found that the two curves, the *non i* and the *i*, were in planes at right angles to each other, and with their openings in different directions, as shown in Fig. 37. Now notice a peculiarity of this figure. If a horizontal plane be passed either above or below the origin, and a circle with its centre in the z axis be

described so that the two root points lie on its circumference, then whether the plane be above or below the origin, that is whether we get two *non i* roots, or two *i* roots, the two roots are arranged on the circumference of the circle at *equidistant intervals*. This is shown in Fig. 38.

Similarly, in the discussion of the equations of the third degree, $x^3 \quad z$, Chap. V, Fig. 23, we found that the *non i* and *i* curves were arranged as in Fig. 39. This figure shows also two horizontal circles, one above and one below the origin, with their centres in the z axis, corresponding to the circles described in the preceding paragraph in the case of equations of the second degree.

This circle will have its centre on the z axis, as can be shown as follows: One root of $x^3 = 1$ is $+1$. Dividing out $x-1$, and solving the resulting quadratic, we find the other two roots to be

$$-\frac{1}{2}+\frac{\sqrt{3}}{2}i, \quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i.$$

Plotting these values we get Fig. 40. The points A and C are evidently the same distance from o that D is, and hence a circle with its centre in the z axis will pass through the three root points.

Now let us see if we can find out how the root points are arranged around the circumference.

Suppose the *non i* value of x to be 2, for instance, which gives a corresponding value of 8 for z. Dividing out the factor $x-2$, and solving for the other two values of x we find $x = -1 \pm \sqrt{(3)}\sqrt{(-1)}$.

The horizontal projection of the circle at the height $z=8$, is shown in Fig. 41, as well as the numerical value of the co-ordinates of the root points. If we can find the angle AOB, we can locate the root point A. From plane trigonometry,

$$\tan AOB = \frac{AB}{BO} = \frac{\sqrt{(3)}}{1} = \sqrt{(3)},$$

and similarly tangent $BOC = -\sqrt{(3)}$. Hence, $AOB = BOC = 60°$, and the root points A, C, D, must be distributed around the circumference at *equidistant intervals*, just as in the case of equations of the second degree.

In the same way, by solving $x^4 = z$ for $x = 2$, we get $x = \pm 2, \pm 2\sqrt{(-1)}$. Plotting these values, we find again, that they are distributed around the circle at *equidistant intervals*.

*Solving the equation $x^3 = -1$, we find

$$x = -1, \quad \frac{1}{2} \pm \frac{\sqrt{(3)}}{2}\sqrt{(-1)},$$

and again the root points are arranged at equidistant intervals.

†Solving $x^4 = -1$, we find

$$x = \frac{1}{2} \pm \frac{\sqrt{(-1)}}{2}, \quad -\frac{1}{2} \pm \frac{\sqrt{(-1)}}{2},$$

*$x^3 + 1 = 0 = (x+1)(x^2 - x + 1)$, whence $x = -1$, $\dfrac{1 \pm \sqrt{(-3)}}{2}$

† $x^4 + 1 = 0$ whence $x^2 = \pm\sqrt{-1}$, $x = \pm\sqrt{-\sqrt{-1}}$. Solving this by the formula for quadratic surds,

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

in which $a = 0$, $b = -1$, we get $x = \dfrac{\pm 1 \pm \sqrt{(-1)}}{2}$.

and again the root points are arranged at equal intervals around the circumference of the circle.

Another important fact is now brought into view, namely, that the root points are arranged symmetrically about the x axis, that is, where there is an even number of root points, the x axis passes midway between the points taken in pairs. When the number is uneven, the x axis passes through one of them, and midway between the others in pairs.

These peculiarities will be found to hold for all values of the exponent of x. Hence the

RULE.—*Describe a circle with a radius equal to the n^{th} root of a. Divide the circumference into n equal parts. Draw the $+x$ axis, or the $-x$ axis through one of these points according as $+a$ or $-a$ satisfies the equation: if neither $+a$ nor $-a$ satisfies the equation, draw the x axis midway between two consecutive points. The abcissas and the ordinates of the different points will be the non i and the i parts of the roots.*

If it is desired to get the roots more exactly, their values can be calculated from a table of natural sines and cosines. Thus,

EXAMPLE.—Required the root of $x^{12} = -1$.

Dividing the circumference into 12 parts, we have each part equal to $30°$. Drawing the x axis, so that these points may be symmetrical about it, we find that the angles between the $+x$ axis and the different points are $15°$, $45°$, $75°$, $105°$, $135°$, $165°$, $195°$, ... Looking in a table of natural sines and cosines for these different angles, and adding the cosine to the sine after we

have multiplied the sine by $\sqrt{(-1)}$ to indicate its direction, we find

$$x = \begin{cases} \cos.\ 15^\circ \pm i \sin.\ 15^\circ. = \dfrac{1+\sqrt{3}}{2\sqrt{2}} + i\dfrac{1-\sqrt{3}}{2\sqrt{3}} = .9659 \pm .2588\ i \\ \cos.\ 45^\circ \pm i \sin.\ 45^\circ. = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}} = .7071 \pm .7071\ i \\ \cos.\ 75^\circ \pm i \sin.\ 45^\circ. = \dfrac{1-\sqrt{3}}{2\sqrt{2}} + i\dfrac{1+\sqrt{3}}{2\sqrt{2}}\ .2588 \pm .9659\ i. \\ \text{etc., etc.} \end{cases}$$

The double sign ($\pm$) is placed before the imaginary part at once since the root points occur in pairs, and we need not calculate the roots for the points occurring in the 3d and 4th quadrants.

If the example had been $x^{12} = -a$, each one of the roots already found should be multiplied by $a^{\frac{1}{12}}$.

EXAMPLE.—Required the roots of $x^4 = -1$.

One-fourth of 360° is 90°. Passing the x axis symmetrically through the circle, we find that the angles which the points make with $+x$ axis, are 45°, 135°, etc., whence

$$x = \begin{cases} \cos.\ 45^\circ \pm \sin.\ 45^\circ.\sqrt{(-1)} \quad \dfrac{1}{\sqrt{2}} \pm \dfrac{\sqrt{(-1)}}{\sqrt{2}} \\ \cos.\ 135^\circ \pm \sin.\ 135^\circ.\sqrt{(-1)} \quad -\dfrac{1}{\sqrt{2}} \pm \dfrac{\sqrt{(-1)}}{\sqrt{2}} \end{cases}$$

GENERAL FORMULA FOR THE Nth ROOTS OF +1.

Let us take the roots of $x^3=1$. Applying the rule, *i. e.*, dividing the circumference of the circle into 3 equal parts and drawing the x axis through one we get Fig. 42. Now the different angles in the figure are evidently

$$\frac{0\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{6\pi}{3},\ \text{etc.}$$

and their sines and cosines are the required abscissas and ordinates, the angles from $6\pi \div 3$ on, giving the same abscissas (sines) and ordinates (cosines) over again, and therefore the same roots in repetition. The same reasoning will apply whatever the degree of the equation.

Now these angles can all be comprehended in the general formula

$$\frac{2\lambda\pi}{n},$$

where λ takes in succession all integral values beginning with zero, and n is the exponent of x in the given equation.

But the abscissas or cosines are the real parts of the roots, and the ordinates or sines the imaginary parts, and hence the roots are given by the general formula

$$x = \cos\frac{2\lambda\pi}{n} + \sqrt{(-1)}\,\sin\frac{2\lambda\pi}{n} = \sqrt[n]{(-1)}$$

where λ takes in succession the values of 0, 1, 2, 3, etc.

GENERAL FORMULA FOR THE Nth ROOTS OF -1.

Take the equation $x^5 = -1$. Applying the rule, we get Fig. 43. In this figure, the first root point, A, corresponds to the angle

$$\frac{\pi}{5}$$

and the second root point, B, to the angle

$$\frac{\pi}{5}+\frac{2\pi}{5},$$

and so on, hence our series of angles are

$$\frac{\pi}{5},\quad \frac{\pi}{5}+\frac{2\pi}{5},\quad \frac{\pi}{5}+\frac{4\pi}{5},\ \text{etc.}$$

which are comprehended in the general formula

$$\frac{1+2\lambda\pi}{n}$$

where λ takes all integral values in succession beginning with zero, and n is the exponent in the given equation.

Hence, just as before

$$x=\sqrt[n]{(-1)}=\cos\frac{1+2\lambda\pi}{n}+\sqrt{(-1)}\sin\frac{1+2\lambda\pi}{n}$$

where λ takes in succession the values 0, 1, 2, 3, etc.

CHAPTER IX.

STURM'S THEOREM.

STURM'S THEOREM is a theorem by means of which we are enabled to find the *number* and *situation* of the real roots of any numerical equation with a single unknown quantity, real and rational co-efficients, and without equal roots.

THE STURMIAN FUNCTIONS of $f(x)=0$ (which has no equal roots*) are functions obtained by treating $f(x)$ and its first derived polynomial $f'(x)$, as in the process of finding their highest common divisor, except that in the process we must not multiply or divide by a *negative* number, and the signs of the several remainders must be changed before they are used as divisors. *These remainders with their signs changed are the Sturmian Functions*, and they are represented by $f_1(x), f_2(x)$, etc.

STURM'S THEOREM. In the series of functions, $f(x)$, $f'(x), f_1(x), f_2(x), f_3(x)$, etc., when $f(x)=0$ has no equal roots, if x be supposed to pass through all possible real values, that is, to vary continuously from $-\infty$ to $+\infty$, there will be no change in the number of variations and permances in the signs of the functions, except when x passes through a root of $f(x)=0$: and when it does pass through such a root, there will be a loss of one variation, and only one.

PROOF.

1. *No two consecutive functions can vanish, i. e., become zero, for the same value of x.*

* If the given equation has equal roots, that fact will be shown in the process of finding the Sturmian Functions; the last remainder will be zero. In this case, all, or all but one of the equal roots must be divided out before proceeding to find the Sturmian Functions.

In the process of finding the Sturmian Functions from $f(x)$ and $f'(x)$, let the several quotients be represented by q, q', q'', etc., whence, by the principles of division, we have

$$f(x)=f'(x)\cdot q-f_1(x), \qquad (1)$$
$$f'(x)=f_1(x)\cdot q'-f_2(x). \qquad (2)$$
$$f_1(x)=f_2(x)\cdot q''-f_3(x), \qquad (3)$$
$$f_2(x)=f_3(x)\cdot q'''-f_4(x), \qquad (4)$$
$$f_3(x)=f_4(x)\cdot q^{\text{iv}}-f_5(x), \qquad (5)$$
etc., etc., etc.

Now, if possible, suppose that some value of x, as $x=a$, renders two consecutive functions as $f_2(x)$ and $f_3(x)$ each o; *i. e.*, the graphs cross the x axis at the same point. But, in equation (4), if $f_2(x)$ and $f_3(x)$ each equal zero, then $f_4(x)=0$. In the same way since $f_3(x)$ and $f_4(x)$ each equal zero, $f_5(x)=0$, and so on down the series of functions. But the last function, that is the last remainder arising from the process of finding the different functions, cannot be zero. Therefore, our hypothesis that two consecutive functions can vanish for the same value of x is incorrect.

Geometrically this means that no two of the graphs of the different functions can cross the x axis at the same point. See Fig. 44.

This statement does not hold when there are equal roots, as will be seen by comparing Figs. 27–31.

2. *When any one of the functions, except $f(x)$, vanishes for a particular value of x, the adjacent functions have opposite signs for this value*, *i. e.*, when the graph

of any function except the primitive crosses the x axis, the two adjacent graphs will at that point lie on different sides of the x axis.

Reference to Fig. 44 will illustrate this. It can be shown analytically as follows: If $f_3(x)$, for example, is zero for any paticular value of x, equation (4) shows that $f_2(x) = -f_4(x)$, and so for any other function.

3. *When any value of x, as $x=c$, causes any function except $f(x)$ to vanish, the number of variations and permances of the signs of the functions is the same for the preceding and succeeding values of x, i. e., for $x=c-h$, and $x=c+h$, h being very small.*

Preliminary conditions.

1°. Some function, say $f_3(c)=0$.

2. No two consecutive functions can vanish at the same time.

3. When a function vanishes the two adjacent ones have opposite signs.

4. A function cannot change its sign without vanishing.

5. A function, not having equal roots, must change its sign when it passes through zero.

6. A function having equal roots does not change its sign when it passes through zero.

In the following tables the first column gives the number which is substituted in the functions at the heads of the other columns; the remaining columns contain the results of these substitutions. The small figures accompanying the plus and minus signs refer to the preliminary conditions which give the reasons for the sign in question.

In case the function which vanishes has not equal roots, we get

		f_2	f_3	f_4		f_2	f_3	f_4
	$c-h$	$+_{4}$	$\pm_{5}$	$-_{4}$		$-_{4}$	$\pm_{5}$	$+_{4}$
either	c	$+_{2,3}$	0_{1}	$-_{2,3}$	or	$-_{2,3}$	0_{1}	$+_{2,3}$
	$c+h$	$+_{4}$	$\mp_{5}$	$-_{4}$		$-_{4}$	$\mp_{5}$	$+_{4}$

Where the sign $\pm$ occurs, it indicates that the result may be either $+$ or $-$, but the two upper signs must go together, and the two lower signs must go together.

The order of the figures referring to the preliminary conditions also indicates the order in which the results of the substitutions are written down.

If the function which vanishes has equal roots, we get in a similar manner the following tables.

		f_2	f_3	f_4		f_2	f_3	f_4
	$c-h$	$+_{4}$	$\pm_{6}$	$-_{4}$		$-_{4}$	$\pm_{6}$	$+_{4}$
either	c	$+_{2,3}$	0_{1}	$-_{2,3}$	or	$-_{2,3}$	0_{1}	$+_{2,3}$
	$c+h$	$+_{4}$	$\pm_{6}$	$-_{4}$		$-_{4}$	$\pm_{6}$	$+_{4}$

In any one of these cases, counting up the variations before the root, and after the root, i. *e.* for $c-h$ and $c+h$, we find that no variations are lost in passing from $c-h$ to $c+h$. This can be verified from the figure by drawing a vertical line just before and just after the

root, and counting up the variations in the signs of the functions, being careful to take them in order.

4. *Whenever x passes through a root of $f(x)$ one variation is lost.*

Preliminary conditions.

1. $f(r)=0$, r being the value of one root of the primitive.
2. No two consecutive functions can vanish at the same time.
3. A function cannot change its sign without vanishing.
4. $f(r+h)$ and $f'(r)$ have the same sign.
5. $f(x)$ must change its sign when it passes through zero, because it has no equal roots.

The explanation for the tables is the same as for the tables in the preceding section. Making the substitutions, we get

either

	f	f'
$r-h$	— 5	+ 3
r	0 1	+ 2
$r+h$	+ 4	+ 3

or

f	f'
+ 5	— 3
0 1	— 2
— 4	— 3

Counting up the variations, we see that one is lost every time $f(x)$ passes through zero; hence the theorem.

Verify this by the figure.

Since a variation is lost every time we pass over a root of $f(x)$ and at no other point, the number of variations lost must equal the number of roots passed over. Hence the theorem. Q. E. D.

Sturm's Theorem is used for finding the first figures of the roots, that is, the approximate positions of the roots, and then the remaining figures are found by Horner's Method which is described hereafter.

EXAMPLE. Find the number and position of the real roots of $f(x)=x^5-13x^4+53x^3-49x^2-110x+150$.

Forming the first derivative and proceeding to find the different Sturmian functions, we get

$$f'=5x^4-52x^3+159x^2-98x-110$$
$$f_1=73x^3-666x^2+1737x-1160.$$
$$f_2=1941x^2-14212x+22535$$
$$f_3=x-5$$
$$f_4=0$$

Since the last function reduces to zero, the primitive has equal roots, and since the last divisor (*i.e.* the last function but one) was $(x-5)$, the equal roots must be 5, 5. We could take out both of these roots by dividing the primitive by $(x-5)^2$, and then proceed to find new Sturmian functions, but since we have already found our functions, with the factor $(x-5)$ in them, it is easier to divide out this factor from all the functions already found.

The easiest way to do this would be of course by synthetic division. Doing this we get

$f = x^4 - 8x^3 + 13x^2 + 16x - 30.$

$f' = 5x^3 - 27x^2 + 24x + 22$

$f_1 = 73x^2 - 301x + 232$

$f_2 = 1941x - 4507$

$f_3 = 1$

The graphs of these functions are shown in Fig. 44. Substituting different values of x in these, we get in the different functions

For $x = -\infty$ the signs $+ - + - +$, 4 variations.

" $x = +\infty$ the signs $+ + + + +$, 0 "

Hence, since 4 variations are lost there must be 4 real roots.

For $x = 0$ we get the signs $- + + - +$, 3 variations. Hence, since we have lost one variation between $-\infty$ and 0, there must be one negative root.

Substituting -1, and -2, we find that the variatiou was lost between these points, and hence one root is between -1 and -2.

In the same way we find that one variation is lost between 1 and 2, which locates another root.

In continuing our substitutions, we find that 3, and 5 reduce the primitive to zero, and are therefore exact roots.

We have now located all the roots, that is, found their position on the x axis, so far as the integral portion is considered.

We could proceed to locate them more closely by substituting successively, for instance in the case of the

root between 1 and 2, 1.1, 1.2, 1.3 and so on. The loss of variations would tell between which tenths the root was located.

This would however be tedious and wearisome, and a much better and shorter method for the figures coming after the decimal point is that of Horner.

HORNER'S METHOD.

Horner's method consists in calculating the coefficients of a series of equations in each of which the roots are less than the roots of the preceding equation by a known amount.

The roots of the *second* are less than those of the *first* by the number of *units* in the required roots.

The roots of the *third* are less than those of the *second* by the value of the first decimal place of the required root.

The roots of the *fourth* are less than those of the *third* by the value of the second decimal place of the required root, and so on.

GEOMETRICAL INTERPRETATION.—Each of these transformations is equivalent to moving the origin along the x axis an amount equal to the corresponding diminution of the value of the root. This is shown in Fig. 45, where the *new* y axis is shown in heavy line.

Moving the orgin a units to the right, for instance, gives us a new variable abscissa ξ, which is a units less than the original abscissa x; but does not change the value of the y ordinate. Hence, to diminish the roots by a* units, substitute $(\xi+a)$ for x. And vice versa, to increase the roots, we substitute $(\xi-a)$ for x.

*a may be any number, whole or fractional.

TO TRANSFORM AN EQUATION INTO A NEW ONE WHOSE ROOTS SHALL BE a* UNITS SMALLER.

We have just found, that to diminish the roots of an expression by a units, we must substitute $(x+a)$ for x. The new x, or ξ as it was there called, will be a units smaller than the old x. The object of this article is to show an expeditious way of getting at the result of that substitution.

On page 20 we found that when $(x+a)$ was substituted for x our expression became

$$f(x+a)=f(x)+f'(x)a+f''(x)\frac{a^2}{\lfloor 2}+f'''x\frac{a^3}{\lfloor 3}+\ldots$$

For simplicity we will here discuss the expression $Ax^3+Bx^2+Cx+D=f(x)$ instead of the general equation of the n^{th} degree.

First compute the different derived polynomials

$$f(a)=Aa^3+Ba^2+Ca+D$$
$$f'(a)=3Aa^2+2Ba+C$$
$$\tfrac{1}{2!}\,f''(a)=3Aa+B$$
$$\tfrac{1}{3!}\,f'''(a)=A$$

By substituting some numerical value for a in these expressions, and adding the results, we get a new expression whose roots will be a units less than the roots of the original equation.

To do this more expeditiously than by direct substitution we proceed as follows:

Evaluate the given expression for a, and then leaving out the last sum (or remainder), evaluate the

* a units does not mean whole units. a may be any number, whole or fractional

remaining sums, together with the first coefficient, as if they were the coefficients of a new equation. Proceed thus, omitting the last sum each time, until only the first coefficient is left.

This operation is shown below.

$$
\begin{array}{lllll}
A & +B & +C & +D & \mid a \\
 & \underline{Aa} & \underline{Aa^2+Ba} & \underline{Aa^3+Ba^2+Ca} & \\
A & Aa+B & Aa^2+Ba+C & Aa^3+Ba^2+Ca+D=f(a) & \\
 & \underline{Aa} & \underline{2Aa^2+Ba} & & \\
A & 2Aa+B & 3Aa^2+Ba+C=f'(a) & & \\
 & \underline{Aa} & & & \\
A & 3Aa+B=\frac{1}{2!}f''(a) & & & \\
A & =\frac{1}{3!}f'''(a) & & &
\end{array}
$$

Comparing these results with the derived polynomials which we have already computed, we find that the different sums correspond to the different derived polynomials. Hence these different sums multiplied respectively by the different powers of x, will give the new equation whose roots are a units less.

EXAMPLE. Transform $3x^4-4x^3+8x-12=0$ into another equation, each of whose roots shall be 3 units less than the roots of this.

$$
\begin{array}{cccccl}
3 & -4 & 0 & 8 & -12 & \mid 3 \\
 & \underline{9} & \underline{15} & \underline{45} & \underline{159} & \\
 & 5 & 15 & 53 & 147=f(3) & \\
 & \underline{9} & \underline{42} & \underline{171} & & \\
 & 14 & 57 & 224=f'(3) & &
\end{array}
$$

$$\begin{array}{llll}
3 & 14 & 57 & 224 = f'(3) \\
 & \underline{9} & \underline{69} & \\
 & 23 & 126 = \frac{1}{2}f''(3) & \\
 & \underline{9} & & \\
 & 32 = \frac{1}{3!}f'''(3) & & \\
3 = \frac{1}{4!}f^{iv}(3) & & &
\end{array}$$

Hence the transformed equation is

$$3x^4+32x^3+126x^2+224x+147=0^*$$

2. Transform $3x^4-13x^3+7x^2-8x-9=0$ into another equation whose roots shall be less by 3 than the roots of this. Ans. $3x^4+23x^3+52x^2+7x-78=0$*.

3. Transform $x^5+2x^3-6x^2-10x+8=0$ into another equation whose roots shall be 2 less than the roots of this. Ans. $x^5+10x^4+42x^3+86x^2+70x+12=0$*.

APPLICATION OF HORNER'S METHOD TO THE SOLUTION OF AN EQUATION.

PROPOSITION.—*If* $(a+h)$ *is a root of* $f(x)=0$, *and* h *is sufficiently small with reference to* a,

$$h=-\frac{f(a)}{f'(a)} \text{ nearly.}$$

That is, if $(4.5621=4.5+.0621)$ is the root of an equation $f(x)=0$, then

$$.0621=-\frac{f(4.5)}{f'(4.5)} \text{ nearly.}$$

* Is the x in this equation the same x as in the original equation? How do the two x's differ?

Developing $f(a+h)$ as was done on page 20, we have

$$f(a+h)=0=f(a)+f'(a)h+f''(a)\frac{h^2}{\lfloor 2}+f'''(a)\frac{h^3}{\lfloor 3}+\ldots\ldots$$

Since h is small compared with a, we can neglect the higher powers without much error, and we get

$$h=-\frac{f(a)}{f'(a)}, \text{ nearly.}$$

This is shown geometrically in Fig. 46, the construction of which is evident. From trigonometry, we have

$$\tan\theta=f'(a)=\frac{\text{AB}}{h}=\frac{\text{AC}}{h} \text{ nearly}$$

$$=\frac{f(a)}{h}$$

whence $$h=-\frac{f(a)}{f'(a)} \text{ nearly.}$$

The minus sign is placed in front of the fraction because $f(a)$ and $f'(a)$ have different signs just before the root, and h must be positive. The less the curvature, or the smaller h is, *i. e.* the nearer the point a approaches the root point, the more nearly will this equation be true.

The student will readily see that this proposition means, that having obtained a part of the root, we can obtain the remaining figures of the root more or less correctly by dividing the derivative by the primitive,

(both being evaluated for the part of the root already found) and changing the sign of the result.

It is the application of this principle that constitutes Horner's Method. We get our start by calculating the integral figures of the root by Sturm's Theorem, and then apply this principle.

HORNER'S RULE.

RULE.—1. Put the equation in the form $Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots\ldots Mx + L = 0$, in which the coefficients A, B, $C \ldots\ldots L$, if not integral, are expressed exactly in decimal fractions.

2. Find the number and situation of the positive real roots by Sturm's Theorem, determining one or more (usually two) of the initial figures: or by trial.

3. Write the coefficients in order with their proper signs, being careful to supply with o's the places of coefficients of missing terms, if the equation is not complete. Taking the initial figures of one of the roots already found, operate on these coefficients so as to obtain the coefficients of the transformed equation whose roots shall be less by the portion of the root already found.

4. Having found these coefficients of the transformed equation, if the coefficient $f'(a)$ of the first power of the unknown quantity in this equation, and the absolute term, $f(a)$, have unlike signs, divide the former by the latter, and the first figure of the quotient will be (approximately) the next figure of the root. If these functions have like signs, more figures of the root must be found by Sturm's Theorem or by trial, before proceeding to apply this process of transformation.

5. Having found a figure of the root by dividing $f(a)$ by $f'(a)$, annex it to the portion of the root already found, and operate on the coefficients of the last (the transformed) equation as they stand, to produce the coefficients of the next transformed equation, *i. e.*, the one whose roots shall be less than those of the last, by the last figure of the root, and less than those of the given equation, by the entire portion of the root now found. Having found these coefficients, divide the absolute term by the coefficient of the first power of the unknown quantity, if their signs are unlike, and so on as before.

6. Proceed in this manner until the root is obtained, or if the root is incommensurable, until as many figures of the decimal fraction are obtained as are desired.

7. In like manner all the positive real roots, or their approximate values may be found. To obtain the negative roots, change the sign of all the terms containing the odd powers of the unknown quantity, and proceed to find the positive roots of this equation as before.

The values thus found will be the numerical values of the negative roots.

GEOMETRICAL INTERPRETATION OF THE ABOVE RULE.

SEC. 4. By reference to Fig. 32, it will be seen that the moment $f'(a)$ and $f(a)$ have the *same sign* it is an indication that we have passed over the root point and that the last figure of the root is too large. We must accordingly reduce this last figure and try our transformation over again until we get a $f'(a)$ and $f(a)$ which are not alike in sign.

If *both* $f'(a)$ and $f(a)$ *change sign*, so that they still are unlike in sign, it shows that we have passed over the nearest root point and into the vicinity of the next one, as shown in Fig 47. In such case there are two roots which have the same initial figure or figures, *e. g.* one may be 23.56+ and the other 23.59+. To obtain the less of the two roots, take the *largest* figure which will not cause either function to change its sign; and for the larger of the two roots take the *smallest* figure which will cause both $f'(a)$ and $f(a)$ to change sign.

In other words to get the smaller root pass to the line A in Fig. 47, which lies near the root point, and to get the larger of the two roots pass to the line B in Fig. 47, which lies beyond one root point, but not as far on as the other root point.

If $f(a)$ *change sign and* $f'(a)$ *become zero*, then we have passed over the root point and have reached the point B in Fig. 47* and we must proceed to find the next figures as in the case where both $f(a)$ and $f'(a)$ changed signs.

If the *first transformation* (*i. e.*, into an equation whose roots are less by the *initial* figure) makes $f'(x)=0$, it shows that the initial figure of the root is at the point B, Fig. 47*, and that the initial figure (a) happens to be a root of $f'(x)$.

If the quotient of $f(a)$ by $f'(a)$ is large, it shows that we are still comparatively a long way from the root point, and that the remaining part of the root is large compared with the part already found. If the quotient is *very* large we are not only far from the root point but the

value of $f'(a)$ at that point is numerically small compared with $f(a)$ as shown in Fig. 47*, where $f(a)$ and $f'(a)$ are taken at the point A.

SEC. 5. The transformation is equivalent to moving the origin to the right the amount by which the roots of the transformed equation are less than the roots of the previous equation.

The above discussion has considered positive roots only. By changing the alternate signs of our equation, we change the negative roots into positive roots, and then the above rules apply. Putting a negative sign before the roots thus found, they become the roots of the original equation.

Ex. 1. Find the roots of $x^4-12x^2+12x-3=0$.
Ans. -3.9073, 0.44327, 0.606018, 2.85808.

Ex. 2. Find the roots of $x^3-3x=1$.
Ans. -0.34729, -1.53208, 1.87938.

Ex. 3. Find the roots of $x^3-6x-2=0$.
Ans. -0.33987, -2.26180, 2.60167.

Ex. 4. Find the roots of $3x^4-4x^2+2x-1000=0$.
Ans. -4.3606, 4.3424.

Ex. 5. Solve $2x^2-5x+3y-2xy+4x^2-12=0$; $4y^2-3x-2y-5=0$. Ans. $y=1.12048, 1.92595, -0.40623, -1.3902$. $x=-0.7396, 1.9951, -1.1757, 1.837$.

Ex. 6. Find the fifth root of 2628.6674882643.
Ans. 4.83.

Ex. 7. Find the cube root of 3. Ans. 1.44224.

CHAPTER X.

COMPLEX QUANTITIES.

All quantities of the form $a+\sqrt{(-1)}b$, where a and b are real, are called *complex quantities.*

To represent them graphically, we adopt the same convention in regard to $\sqrt{(-1)}$, or i as we shall denote it, as before; that is, we will lay off real quantities along one axis and pure imaginaries along an axis perpendicular to it, as shown in Fig. 48. This is called *Argand's Diagram.* The quantity $a+ib$ will thus locate in the plane of the axes, a point P whose rectangular co-ordinates are a and b.

Any complex number $a+ib$ can be written in the form

$$\sqrt{(a^2+b^2)}\left(\frac{a}{\sqrt{(a^2+b^2)}}+i\frac{b}{\sqrt{(a^2+b^2)}}\right)=a+ib.$$

The expressions $\frac{a}{\sqrt{(a^2+b^2)}}$ and $\frac{b}{\sqrt{(a^2+b^2)}}$ may be taken as the sine and cosine of some angle φ, since they satisfy the equation $\cos^2\varphi+\sin^2\varphi=1$. If we put $r=\sqrt{(a^2+b^2,)}$ the complex number may be written $a+ib=r\,(cos\,\varphi+i\,sin\,\varphi)$. This might have been derived directly from the figure since $r\,cos\,\varphi=a$, and $r\,sin\,\varphi=b$.

$r=OP=\sqrt{(a^2+b^2)}$ is called the *modulus* of the complex number $a+ib$, and the angle φ is called the *amplitude.*

If a complex number vanishes, its modulus vanishes, for that is the only way in which the point P can be brought to the origin; and conversely, *if the modulus vanishes, the complex number vanishes.*

ADDITION OF COMPLEX QUANTITIES.—We can represent the sum of $a+ib$ and $x+iy$ by first representing $a+ib$, which gives the point P, Fig. 49; then using this point as a new origin, plot $x+iy$, which gives a new point Q. OQ is the modulus of the sum, and ψ the amplitude. In this way we can add any number of complex quantities, and the sum will be represented by a point in the plane. If the sum is *zero*, the modulus must be zero and the point must coincide with o, the origin. Hence the different moduli of the quantities added will form a closed polygon with one vertex at the origin, as for example the polygon OPQO.

It is easily seen frem the figure, that *the modulus of the sum of complex quantities is less than the sum of the moduli.*

SUBTRACTION OF COMPLEX QUANTITIES is a special case of addition.

MULTIPLICATION OF COMPLEX QUANTITIES.—Let $r(\cos\varphi + i\sin\varphi)$ and $r'(\cos\varphi' + i\sin\varphi')$ be two complex quantities. By actual multiplication their product is $rr'\,[\cos\varphi\cos\varphi' - \sin\varphi\sin\varphi' + i\,(\sin\varphi\cos\varphi' + \cos\varphi\sin\varphi')]$. By trigonometry, this may be written $rr'\,[\cos(\varphi+\varphi') + i\sin(\varphi+\varphi')]$.

Therefore *the modulus of the product of two complex quantities is the* PRODUCT *of their moduli, and the amplitude of the product is the* SUM *of the amplitudes.* Consequently the effect of multiplying one complex quantity

by another is to multiply the modulus of the first by the modulus of the second; and to turn the line representing the modulus of the first through the amplitude of the second.

This result is clearly general, for the multiplication of this complex number by a new complex quantity, will evidently give a similar result.

The multiplication of complex quantities can be represented geometrically as follows: Put $rr'=R$. Whence $r':R=1:r$. Let oz, and oz' and oZ represent the moduli as shown in Fig. 50. Connecting z with the $+1$ point, the preceding proportion shows that the triangles zo1, and Zoz' are similar. Hence the

RULE FOR COMPLEX MULTIPLICATION.—Connect the end of one modulus with the $+1$ point, and upon the other modulus construct a similar triangle, the modulus and o1 being homologous lines, and the equal angles being at the origin. The side of this triangle will be the new modulus, and its angle with the x axis the new amplitude. The new modulus is a fourth proportional to o1, and the two given moduli.

This affords an excellent proof of the THEOREM:—If w represent one of the complex cube roots of unity, then $w^2+w+1=0$. In Fig. 42 let OA represent one of the complex cube roots of unity found as described in Chap. VIII. Multiplying it by itself by the rule given above we get OB for the new modulus of w^2. Adding w and w^2, that is OA and OB, by the rules for addition we are carried to the point C. Now adding 1, we are carried back to the point o, *i. e.* $w^2+w+1=0$. Q.E.D.

Since OB is the other complex cube root of unity, as

is easily seen from the figure, we have the additional THEOREM:—The square of one of the complex cube roots of unity equals the other.

This a special case of the general THEOREM:—If a is a root of $x^n=1$, then any power of a is also a root of the same equation.

The proof of this is easily seen when we consider the first rule for multiplication of complex quantities. Each multiplication simply turns the modulus through an angle equal to the amplitude of the root, that is, throws the end of the modulus round to another *root point.*

By comparing Fig. 43 it will be seen that the same theorem holds true for odd powers of roots of $x^n=-1$.

If we put $r_1=r_2=r_3 \ldots\ldots = r_n =1$, we have

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\ldots(\cos\theta_n+i\sin\theta_n)$$
$$=\cos(\theta_1+\theta_2+\ldots+\theta_n)+i\sin(\theta_1+\theta_2+\ldots+\theta_n)$$

This is the most general form of what is known as *DeMoivre's Theorem.* Making $\theta_1=\theta_2=\ldots\theta_n$, we get the more familiar form

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

It is an analytical result of the highest importance.

Involution being a special case of multiplication we are easily led to the following RULE FOR SQUARING A COMPLEX QUANTITY.—*Double the angle and lay off along the side of this new angle a third proportional to* OI *and the given modulus.*

This easily leads to the following RULE FOR INVOLUTION TO n^{th} POWER.—Take n times the given angle and along the side of the new angle lay off r^n found as shown in Fig. 50*.

Since the given modulus is a mean proportional between its square and o1, we easily get the following RULE FOR EXTRACTION OF THE SQUARE ROOT. — *Halve the angle representing the amplitude, and lay off along the bisector a mean proportional between the given modulus and o1.*

We can extract cube roots by trial by reversing the method shown in Fig. 50*, commencing at r^3 and working toward $+1$. If we do not come out at $+1$, try over again.

Hence, RULE FOR EXTRACTION OF CUBE ROOT.— Trisect the amplitude, and lay off the cube root of the modulus found as above.

These rules will be found particularly useful in solving the examples of this and the next chapter.

DIVISION OF COMPLEX QUANTITIES.

The quotient

$$\frac{r(\cos\varphi + i\sin\varphi)}{r(\cos\varphi' + i\sin\varphi')} = \frac{r}{r'}[\cos(\varphi - \varphi') + i\sin(\varphi - \varphi')]$$

found by multiplying numerator and denominator by $\cos\varphi' - i\sin\varphi'$.

Consequently, the *quotient of two complex numbers is a complex number whose modulus is the quotient of the moduli, and whose amplitude is the difference of the amplitudes of the two complex numbers.*

If in the above equation, we make $r = 1$, $\varphi = 0$, we get

$$\frac{1}{r'(\cos\varphi' + i\sin\varphi')} = \frac{1}{r'}[\cos(-\varphi') + i\sin(-\varphi')]$$

$$= \frac{1}{r'} [\cos \varphi' - i \sin \varphi']$$

a useful transformation which will be used hereafter.

Complex multiplication can be shown geometrically as follows: Let $\frac{r}{r'} = R$ whence $r : r' = R : 1$. Let Oz, Oz' and OZ represent the moduli as shown in Figs. 51 and 51*. Connecting Z with the $+1$ point, the preceding proportion shows that the triangles $ZO1$ and zOz' are similar.

Hence the Rule for Complex Division.— Connect the ends of the two moduli and upon the line $O1$, construct a similar triangle, the line $O1$ being homologous with the line representing the divisor.

FUNCTIONS OF A SINGLE COMPLEX VARIABLE.

Suppose $z = x + iy$ and $w = f(z) = f(x+iy) = u + iv$, where f indicates some function, and x, y, u and v are real quantities.

Now as the point determined by z, or as we shall call it, z, moves continuously in the co-ordinate plane, the point w, determined by the co-ordinates u and v, will also move continuously in the co-ordinate plane.

A graphic representation of the function $f(x + iy)$ can be obtained by constructing another diagram for the complex number $u+iv$. Then the continuity of $f(x+iy)$ is expressed by saying that when the graph of the independent variable z is a continuous curve S, the graph of the dependent variable, w, is another continuous curve T.

EXAMPLE.—Let $w = +\sqrt{(1-z^2)}$. For simplicity we will suppose z to have only real values, that is, suppose $y = 0$. The path of the independent variable z is then the x axis. See Fig. 52.

For $z = -10$ for instance, $w = +\sqrt{(-99)} = +9.9\sqrt{(-1)}$; for $z = -1$, $w = 0$; $z = 0$, $w = +1$; $z = 1$, $w = 0$; $z = 20$, $w = +\sqrt{(-399)} \quad +19.9\sqrt{(-1)}$ etc. That is, as z starts on the left and moves towards the right, w starts on the $+v$ axis and moves down to zero, then along the $+u$ axis to $+1$, and then back to zero, and then up the $+v$ axis again. The paths are shown by the heavy lines, and the corresponding positions of z and w by the letters a, a', etc.

EXAMPLE.—Let $w = \frac{1}{z}$. This can written

$$u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.$$

But in this the real and imaginary parts of the two members must be respectively equal, hence

$$u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}$$

whence we easily get $x = \frac{u}{u^2+v^2}$, $y = -\frac{v}{u^2+v^2}$. Now when z moves parallel to the y axis, $x = \text{constant} = c$, and we have $x = c = \frac{u}{u^2+v^2}$ or $u^2+v^2 - \frac{1}{c}u = 0$, which is the curve described by w. By plotting this curve we find that it is a circle with its centre on the u axis and tangent to the v axis, as shown in Fig. 53, and having a

radius $=\frac{1}{2x}$. The graphical construction for two points is indicated in the z plane, the points being afterwards moved over to the w plane.

If z moves parallel to the x axis, $y=$ constant, and we have $y=-\frac{v}{u^2+v^2}$ or $u^2+v^2+\frac{1}{y}v=0$, a circle with its centre on the v axis and tangent to the u axis, as shown in Fig. 54. Let the student construct it graphically, as shown for Fig. 53.

If we write $w=\mathrm{R}(\cos\Phi+i\sin\Phi)$ and $z=r(\cos\varphi+i\sin\varphi)$ then $\mathrm{R}=\frac{1}{r}$ and $\Phi=-\varphi$, since

$$\frac{1}{z}=\frac{1}{r(\cos\varphi+i\sin\varphi)}=\frac{1}{r}(\cos\varphi-i\sin\varphi)$$

$=\frac{1}{r}[\cos(-\varphi)+i\sin(-\varphi)]$. Now as z moves in a circle of radius r around the origin, w must move in a circle of radius $\frac{1}{r}$, through the same angle but in an opposite direction, as shown in Fig. 55. Prove it by graphical construction. The construction lines for one point are shown, the point being afterwards transferred to the w plane. This is not necessary of course except to disentangle the figures.

EXAMPLE.—Let $w=z^2$. This can be written $u+iv=(x+iy)^2=x^2-y^2+2xyi$, whence $u=x^2-y^2$, $v=2xy$. If z moves parallel to the y axis, $x=$constant, and we have after eliminating y, $v^2=4x^2(x^2-u)$. Upon plotting this we find that it is a parabola as shown in Fig. 56.

If z moves parallel to the y axis, then we will get a parabola turned in the other direction.

If R and r are the moduli and Φ and φ the amplitude of w and z, we easily get $R=r^2, \Phi=2\varphi$. That is, as z moves in a circle around the origin, w moves in another circle of r^2 radius and through twice the angle.

If φ is constant, w moves along the second side of the angle 2φ and $\frac{r^2}{r}$ times faster than z moves. See Fig. 57. Verify these figures by graphic construction.

GRAPHIC SOLUTION OF EQUATIONS CONTAINING COMPLEX QUANTITIES.

In chapter II we discussed methods of showing the forms of graphic surfaces by means of vertical contour lines. Our equations in that chapter contained three co-ordinates x, y, z, and since x depends on z and y for its values, the general form will be $x=f(z, y)$. We plotted the vertical contour lines for different constant values of x. Of these contour lines, the one for $x=0$ is a very important one. It is the one cut by the vertical plane zy, Fig. 5. Now this plane cuts the graphic surface in a curve S (zero contour line) every point of which has for its co-ordinates a pair of values that satisfy the equation $x=f(z, y)=0$.

Hence the curve S in this case divides the plane into two parts, such that for any point in either part, $f(z, y)$ is either $+$ or $-$, and the curve S always forms the boundary line between the two parts in which $f(z, y)$ has opposite signs.

This principle can be used for the solution of some complex equations. It will be best illustrated by an

example taken from Chrystal, using however, the letters w, x and u instead of x, z and y respectively, to conform to our previous lettering for complex quantities.

EXAMPLE.—Solve the equation $iz^2+8=0$. Write this in the familiar complex form as follows: $w=iz^2+8$, in which $w=u+iv$, $z=x+iy$. Substituting these values in the given equation, we have $u+iv=i(x+iy)^2+8 =2(4-xy)+(x^2-y^2)i$. Equating the pure reals and the pure imaginaries, we have, $u=2(4-xy)$, and $v=x^2-y^2$, or as they can be written $u=f(x,y)$and $v=\varphi(x,y)$.

Now it must be remembered, that to solve our original equation, we must seek values of z, *i. e.* of x and y, which will make $w=0$, that is, which will make $u=v=0$, for w can only become zero by making $u=v=0$.

Making $u=0$ we have $4-xy=0$, for the equation of the curve S which forms the boundary line between the $+$ and $-$ values of u. This curve is shown in Fig. 58. In the same way, by making $v=0$, we get $x^2-y^2=0$ for the equation of a curve T, the co-ordinates of which make $v=0$. This curve is shown in Fig. 58 by the dotted straight lines. Where these two curves intersect, we will of course have points whose x and y make both u and v and consequently w equal to zero, that is, whose co-ordinates satisfy the given equation. In this case, there will be found on measurement to be for P, $+2+2i$ and for Q, $-2-2i$. It is easy to verify that these expressions are roots of the given function.

Another interesting example is $x^2+2x+a=0$, the same that we discussed in Chapter IV. Putting this equal to $w=u+iv$, and substituting for x, $x+iy$, we get $u+iv=x^2-y^2+2xyi+2x+2iy+a$. Equating the pure

non i and the pure *i* parts respectively, we have for the curves S and T, $u=0=x^2-y^2+2x+a$, $v=0=2(xy+y)$.

The S curve will be found to be a rectangular hyperbola with its centre at $(-1, 0)$ and lying along the y or x axis according as a is greater or less than 1, as shown in Figs. 59-61, where two S curves are shown, the heavier one being for the larger (algebraic) value of a.

The T curve will be found to be the two lines $x=-1$ and $y=0$, the first one represented by the dotted line and Figs. 59-61, and the second one coinciding with the x axis.

The intersection of the two curves S and T gives the roots of the given equation, just as in the previous example. There are two intersections, and hence, two roots. In Fig. 59 they are i, viz., $-1\pm iy$. In Fig. 60 they are equal and equal to -1, and in Fig. 61 they are pure *non i*, being the intersections of S with the x axis.

This example is another illustration of how the complex or i roots gradually merge into the pure *non i* roots, as they travel along the dotted T curve, until they meet the x axis (the other T curve), and then travel out the x axis, right and left. In Fig. 60 the two kinds of roots are shown to coincide, being pure *non i* roots, or complex roots with no i part. It can be written $-1\pm 0i$.

These S curves are the horizontal sections of the hyperbolic paraboloid discussed in Chapter IV, and already shown in pseudo prospective in Figs. 14, 19, 20, where they are represented by the horizontal intersections.

CHAPTER XI.

DOUBLE VALUED COMPLEX FUNCTIONS.

Expressions containing a radical sign, and therefore having two values for each value of the variable are called double valued functions.

The simplest function of this form is $w=\sqrt{z}$, which we will now discuss, first putting it in the form

$$w=\sqrt{z}=\sqrt{[r(\cos\varphi+i\sin\varphi)]}=\pm\sqrt{[r(\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2})]}.$$

For simplicity, we will suppose r constant and equal to 1. Now giving the different values $0, \frac{\pi}{2}, \pi$, etc., to φ we get the value $1, +i, -1, -i$, etc., and $\pm 1, \pm(\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}}), \pm i, \pm(-\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}})$ etc. for z and w respectively. Plotting these we get Fig. 62, the arrow heads indicating the direction of the movement, and the letters the corresponding positions of the two variables, z and w. Now suppose at the point D, φ becomes constant and r variable, it is easily seen that both z and w will move along the radius OD, either inward or outward. When z has arrived at E, $(z=-i\frac{1}{4})$, w will be at E, $[w=\pm\frac{1}{2}(-\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}})]$.

Now, let r become constant, and φ change from $\frac{3\pi}{2}$ back to zero, z and w will evidently describe circular arcs with this present radius back to the initial radius, that is, to the point F. Now let φ be constant and r change from $\frac{1}{4}$ to 1, and the points return to the original point, A.

It will be noticed that the curve described by z has not enclosed the origin, and w assumes its original value when z does. If from point E, φ had changed its value from $\frac{3\pi}{2}$ to 2π instead of to o, z would have resumed its original value, but w would not have done so. The value of w, which started out at $+1$, will now have arrived at -1. This is shown in Fig. 63.

Hence, we see that w returns to its original value or not, according as the curve described by z in returning to its first value, does not or does enclose the origin.

If the z curve intersects the origin, then both the w curves do so also, and we have Fig. 64.

This is the point for this particular function where the two values of w coincide, and is very appropriately called a *branch point.**

Fig. 65 shows the corresponding movements of the two points, z and w, the different kinds of lines corresponding in the two figures, and the starting point being indicated by a black dot. The student can verify these for himself.

As another example take $w=\sqrt{(z-1)}$. Put $z-1=\zeta$ and we have $w=\sqrt{\zeta}$, a form which we have just discussed. But since $\zeta=z-1$, the origin for z must be -1 from that for ζ or 1 unit to the left, giving us Fig. 66 as the corresponding one to Fig. 63, in which, however, r is taken equal to 2 instead of 1. Notice the change in the size of the w curve. Here $+1$ and not o, is the branch point.

We will now take up a somewhat more complicated function, namely, $w=(z-1)\sqrt{z}$. We will put both the

*Verzweigungspunkt, ausnahmspunkt.

z and w curves in the same figure, showing, however, only one of the w curves, and part of the other.

Put $z-1=r(\cos\varphi+i\sin\varphi)$, whence $w=r(\cos\varphi+i\sin\varphi)(1+r\cos\varphi+ri\sin\varphi)^{\frac{1}{2}}$. Also put $1+r\cos\varphi=\rho\cos\psi$ and $r\sin\varphi=\rho\sin\psi$, which gives us $w=r\rho^{\frac{1}{2}}(\cos\varphi+i\sin\varphi)(\cos\psi+i\sin\psi)$. The relation between ρ, r, φ and ψ is shown in Fig. 67. Now let r be constant, that is, let z describe a circle around the $+1$ point. If $r<1$, ($\frac{3}{4}$), we will get Fig. 68. If $r>1$, ($1\frac{3}{4}$), we will get Fig. 69. The scale is the same for both figures. The construction lines for one point are given. A is the position of r for a certain value of φ, B the corresponding position of ρ, C of $\sqrt{\rho}$, D of $r\rho^{\frac{1}{2}}$ or w.

O is a branch point where the two values of w come together, and Figs. 68 and 69 show that w returns to its original value or not according as the z curve does not or does enclose the branch point O, and moreover that $+1$ is not a branch point.

It would be excellent practice for the student to verify these figures by graphical construction, using the rules of Chap. X and a scale about three times as large.

CHAPTER XII.

RIEMANN'S SURFACES.

In the problems of the previous chapter we found that for a certain position of z there might be two values of w, depending upon the location of the curve described by z in regaining its initial position.

Thus in Fig. 63 if z describe a circle around the branch point, w which at the beginning had a pos. value, say $+W$, will when z regains its initial position have the value $-W$, and z must pass through its initial position twice before w can regain its original value. To avoid this ambiguity, we are indebted to a beautiful method devised by the German mathematician, Riemann. Instead of supposing the loops of the z curve to lie in one plane, each complete loop is supposed to lie in a different plane. Thus, in Fig. 70, suppose the z curve there represented to pierce the paper when it gets almost back to its initial point and pass into the next page below, completing its second loop on that page (shown by the dotted line) and then to return to the upper page just as it regains its initial position. The upper loop corresponds to the values of w which begin with $+W$, and the lower loop to those beginning with $-W$, or vice versa. Thus the ambiguity in regard to the value of w is entirely removed. A certain position of z corresponds to a certain value of w, by whatever route z regains that position.

These two planes are supposed to be slit along the x axis from o to ∞, and the right hand edge of the upper plane joined to the left hand edge of the lower plane, and vice versa. This is shown in Fig. 71, very much distorted. The line along which the planes are slit is called the *branching section**; and the surfaces are called *winding surfaces***, (double leaved in the case we are considering) and the whole surface is called a *Riemann's surface****, of the first order in this case.

*Verwachsungslinie, verzweigungsschnitt. **Windungsfläche.

***Riemann'sche Fläche.

The branch point, or end of the branching section, is called a *winding point.*¶

It is easy to see that a z curve which does not enclose the winding point can return to its starting point with only one loop whether it crosses the branching section or not. If it enclose the winding point it must complete two loops, one in each plane before it can return to its starting point.

This agrees with our former method of representation (Chap. XI) where a double loop must be completed by z around the branch point before w could attain its initial value, but only one loop was needed in case it did not enclose the branch point.

The branching section can evidently be any curve, and not necessarily a straight line.

Fig. 71* shows a very irregular double loop made by z in the function $w = \sqrt{(z-1)}$. It will be noticed that it goes around the branching point $+1$, twice, once in the upper Riemann surface, shown by the full line, and once in the lower Riemann surface, shown by the dotted line. The w curve, shown by the heavy line returns to its initial value only after two loops of the z curve. The other w curve, would, if it were necessary to consider it at all be represented by a similar curve revolved 180°, that is, by the part which corresponds to the second loop of z. It will be seen that this is nothing but an Argand Diagram, with the second loop in the lower plane.

Fig. 71** shows the same function when z moves parallel to the axes. As a help to the student the light

¶ Windungspunkt.

line continuations of the w curve show the hyperbolas of which the w curves are limited portions.

If we had a three leaved Riemann surface as in the case of the function $w=(z-1)^{\frac{1}{3}}$ there would be three w curves, all coinciding at the branch point $+1$, and requiring three loops of the z curve to regain their initial values.

On the Riemann surface we should plot only one of these. But if our function were of the form, for instance, $w^3-w=-z$, in which two of the w curves run together at the point $z=\frac{2}{\sqrt{(27)}}$, but the third w curve does not intersect the branching point at all, we could neglect one of the two branching curves and plot only the other, and the third curve.

So far, we have spoken only of functions with one branch point. Functions can have more than one. For instance $w=\sqrt{[(z-a)(z-b)]}$ has two branch points, one at a and one at b. In this case, our branching section, instead of lying between the one branch point and infinity, lies between the two branch points, as shown in Fig. 72. In this case a *closed* curve must go around a branch point twice, as before, or around each branch point once, or both together once, or around none.

The different closed curves are shown in the figure, the boundaries of the limited surfaces there shown being considered as one case of a closed curve. A special case for $w=\sqrt{[(z-1)(z-2)]}$ is shown in Figs. 72* and 72**, the branching points being indicated by small dots, connected by the branching section.

In the case of three winding points, a branching

section connects two of them and the other branch section extends from the third winding point to infinity. The branching section connecting two winding points evidently performs the same office as two extending to infinity, one from each winding point.

If there are three planes so connected at a winding point that a curve enclosing the winding point must have three loops before it can close, then we have a Riemann's surface of the second order, and so on. Fig. 72*a* shows a winding surface of the second order. Fig. 72*b* shows a fully delimited surface cut from a winding surface of the first order with three winding points, and the part that remains after the delimited surface is taken out. The portions belonging to the lower plane are shaded lightly, and the heavy shading indicate absence of any surface.

Fig. 72*c* does the same thing for two winding points, and Fig. 72*d* for one winding point. A careful study of these figures will well repay the student.

A *closed curve* is one which is continuous, returning into itself without crossing itself.

A closed curve which is other than an enlargement of the boundary of the surface, is said to *fully delimit** a surface when it is impossible to pass from one side of the curve to the other without crossing the curve itself.

For example, any closed curve on an ordinary plane (Fig. 38); any circle on a sphere which thus divides the sphere surface into two calottes; two meridian circles on a torus; a closed curve around one winding point (Fig. 72*d*).

*Vollständig begrenzen.

A closed curve does *not* fully delimit a surface, when it is possible to pass from one side of the curve to the other without crossing the curve: *e. g.* one meridian curve on a torus; or a meridian and an equatorial curve on a torus; a single loop around two winding points, there being four altogether (see Fig. 73, the arrow-headed line being the path from one side to the other of the closed curve).

The dotted line in Fig. 74 does not fully delimit a portion of the surface, since it is a mere enlargement of the inner boundary.

A *simply connected** surface is one in which *every* closed curve fully delimits a portion of the surface. The test is that if the surface be cut through along the curve, one part of the surface can be entirely separated from the other. For example,—an ordinary plane; a winding surface of the first order with one winding point; the surface represented in Fig. 73 if the arrow heads be connected.

A *doubly connected* surface is one in which *one* and *not more* than one closed curve can be drawn which does not fully delimit a portion of it. Hence, one of two closed curves, if not both, will fully delimit a portion of the surface. For example,—the portion of an ordinary plane enclosed between two closed curves (Fig. 74); a winding surface of the first order with two winding points: the same surface with four winding points, two of them being within a closed curve (see Fig. 73).

A *triply connected* surface is one in which two and no more closed curves can be drawn without fully delimit-

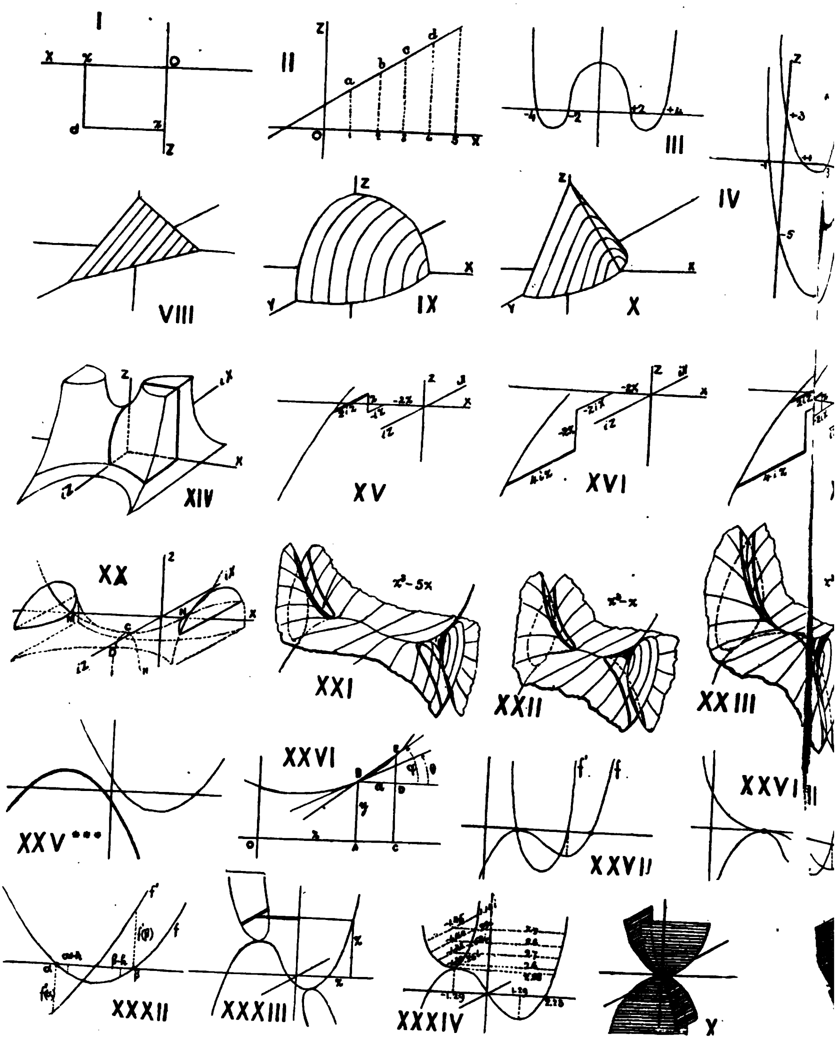
* Einfach zusammenhängenden.

ing a portion of the surfaces. And so on for multiply connected surfaces.

Simply connected surfaces and the conversion of multiply connected into simple, play an important part in the Theory of Functions. The investigation of functions of the $\frac{1}{2}$ degree depends upon the Riemann surfaces of the first order; of the $\frac{1}{3}$ degree upon surfaces of the second order and so on.

The number of winding points depends upon the degree of the function under the radical sign. The third and fourth degree expressions for radix of $\frac{1}{2}$ lead to what are known as Elliptic Functions. Higher degree expressions lead to hyperelliptic functions.

Further investigation in this line must be preceded by a knowledge of the Integral Calculus and the Function Theory.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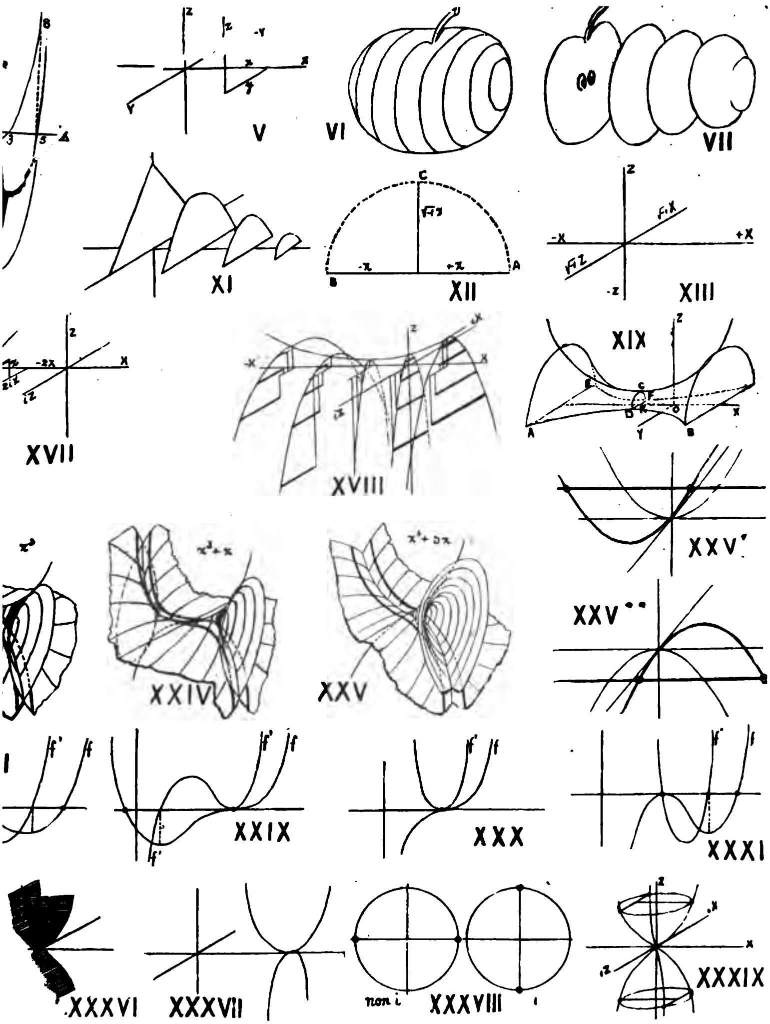
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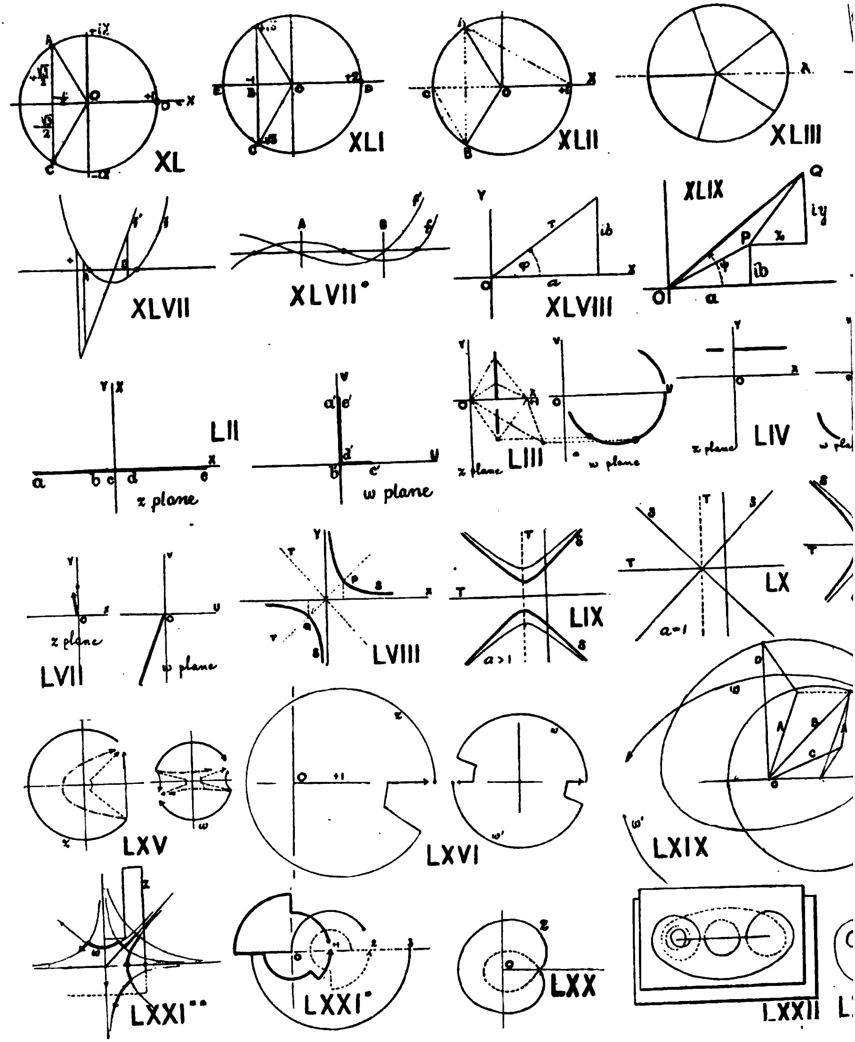
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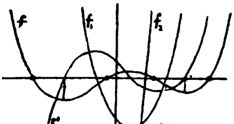
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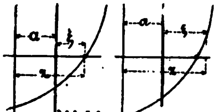
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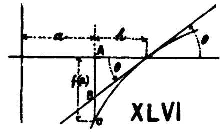
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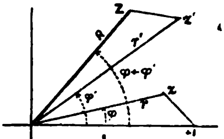
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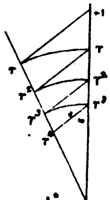
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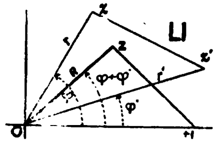
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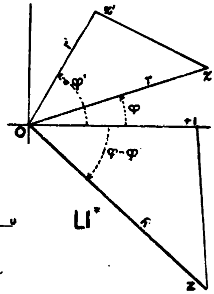
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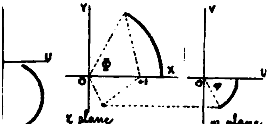
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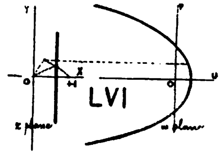
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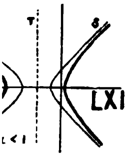
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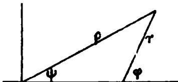
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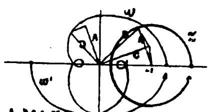
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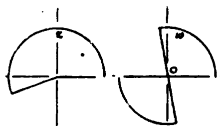
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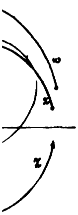

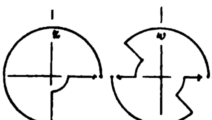
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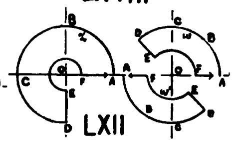
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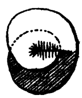
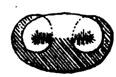
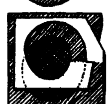
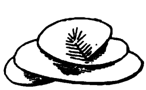





LXXIId


LXXIIc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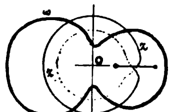
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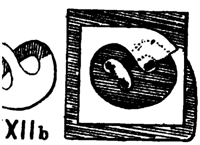
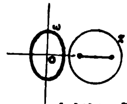

XIIb


LXXI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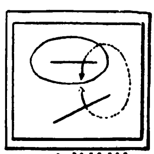
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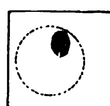
LXXIV